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A System to Measure Thermal Resistivity of Soils

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A SYSTEM TO MEASURE
THERMAL RESISTIVITY OF SOILS

BY

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A SYSTEM TO MEASURE

THERMAL RESISTIVITY OF SOILS

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable for meeting the thesis requirements for this degree. Acceptance of this thesis does not imply that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Adviser   Date

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A SYSTEM TO MEASURE

THERMAL RESISTIVITY OF SOILS

Abstract

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Under the supervision of Professor Robert A. Higgins

A transient heat-flow, cylindrical-probe method for measuring thermal resistivity of soils *in situ* is investigated. The theory of conduction of heat in a medium containing a cylindrical heat source is studied in detail. Effects of finite probe radius, length, and thermal conductivity are investigated and minimized. The effect of thermal contact-resistance at the probe surface is also studied. Errors in measurements of samples of finite dimensions are considered.

An electronic system which uses the cylindrical-probe method to measure thermal resistivity of soils is described. Results are given for the use of the system in the measurement of thermal resistivity of a fine white sand. Thermal resistivity was found to vary significantly with moisture content.

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in the soil, and therefore upon the thermal conductivity of the soil [Shaw and Baver 1939b]. Various systems for measuring moisture content of soils by methods which depend upon thermal conductivity have been presented in the literature [Bouyoucos 1950; Bloodworth and Page 1957; Cornish, Laryea, and Bridge 1973].

An important application of soil moisture data is in irrigation practices. Knowledge of the amount of moisture in the soil would help the farmer or rancher who irrigates his land to know when and how much to irrigate. Soil moisture content data is also of interest in the planning of many major constructions such as roads, dams, large buildings, etc.

For either application, a field method of determining thermal conductivity (or resistivity) is desirable. Laboratory methods require sampling the soil. The resulting disturbance inherently introduces errors in measurements.

1.2 Purpose of thesis

The objective of this thesis is to propose a practical method to measure thermal resistivity of soils. To allow field measurements, the measuring system must be portable. That is, it must be easily moveable and must have its own (portable) power supply.

The system should also require a minimum of technical knowledge from its operator. The system should measure thermal resistivity directly, rather than supplying data which must be interpreted by the use of charts, tables, graphs, etc. Application of the measurements to determination of soil moisture may, however, require charts, tables, or graphs to account for such variables as mineral content and density of soils.

The system should be available at a cost that is not prohibitive. The system is obviously of no use if the potential user cannot afford it.

Above all, the system must operate reliably over a large range of climatic conditions. Temperature variations from below freezing to over 100° F will affect the thermal conductivity of the soil, but should not appreciably affect the operation of the measuring system.

1.3 Brief overview of thesis

To design a system to measure a thermal property such as thermal conductivity or its inverse, thermal resistivity, one must first have a knowledge of the theory of heat transfer. Although soils contain elements of all phases, solid, liquid, and gas, the theory presented in this thesis is valid for heat transfer in solids only. It is only in solids that the transfer of heat by movement of particles of the heated body (convection) can be assumed negligible. The theory of heat transfer is also simplified by assuming no radiation of heat. The only means of heat transfer considered is conduction.

While convection admittedly can occur in soils, it can be minimized by limiting any temperature gradients which may be caused by a measurement, and by limiting the interval of time over which any temperature gradient may occur.

In Chapter 2, the partial differential equation of heat conduction is derived. The solution for the temperature of a medium containing an infinite line source of heat is then found. A method of measuring thermal resistivity (or conductivity) using an infinite line source of heat is presented in §2.3. Constraints on the source of heat are considered, one by one, in the later sections of Chapter 2, until the solution for temperature of a cylindrical probe of finite radius and finite length contained within a finite medium is considered in §2.8. It is found that if the probe and measurement conditions mentioned in §2.9 are met, the method of measuring thermal resistivity still applies.

An electronic system which uses the method of §2.3 to measure thermal resistivity is described in Chapter 3. The system, which can be operated from two 9-V batteries and a 12-V auto battery, is portable. Thermal resistivity measurements are read directly. The system can be calibrated to display values in any system of units. With the current fluctuation of prices of TTL integrated circuits, it is difficult to estimate the cost of the system. The cost of all of the parts contained in the system is under $100 at current prices.

The thermal-resistivity-measuring system was used to measure thermal resistivity of a fine silica sand. Measurements were taken at various moisture contents. Results can be found in Chapter 4 and Appendix III.

CHAPTER TWO

THE TRANSIENT FLOW METHOD OF MEASURING THERMAL CONDUCTIVITY

2.1 Introduction and theory

In this chapter the flow of heat in an infinite, isotropic, homogeneous solid is considered. In other words every point within the solid has identical physical properties. When a point is heated, the heat flows equally well in all directions, rather than favoring certain directions as in crystalline and anisotropic solids. In this chapter the medium in which heat is assumed to flow is limited to solids because heat flow in solids is due to *conduction* alone. The transfer of heat by *convection* is negligible.

A background theory of heat conduction is presented in this section. The application of this theory to a generalized problem is found in §2.2. A method of measuring thermal conductivity is given in §2.3. Later sections of this chapter apply more stringent conditions, one by one, to the generalized problem of heat conduction of §2.2. In each case, it is shown that the method of §2.3 for measuring thermal conductivity applies.

2.1.1 Notation and units

r = radial coordinate (cm)

θ = angular coordinate (radians)

z = linear coordinate (cm)

t = time (sec)

ℓ = length of probe (cm)

b = outer radius of cylindrical probe (cm)

a = inner radius of hollow probe (cm)

q = heat per unit length per unit time supplied by probe (cal/cm-sec)

T = temperature of solid surrounding probe (°C)

T_p = temperature of probe (°C)

$T_{p,a}$ = temperature of inner surface of hollow probe (°C)

$T_{p,a,0}$ = temperature at the midpoint of the inner surface of hollow probe of finite length (°C)

K = thermal conductivity of solid surrounding probe (cal/cm-sec-°C)

K_p = thermal conductivity of probe (cal/cm-sec-°C)

H = thermal conductivity per unit length at $r = b$ (cal/cm^2-sec-°C)

ρ = density of solid surrounding probe (g/cm^3)

ρ_p = density of probe (g/cm^3)

M_p = mass per unit length of probe (g/cm)

c = specific heat of solid surrounding probe (cal/g-°C)

c_p = specific heat of probe (cal/g-°C)

$\alpha = K/\rho c$ = thermal diffusivity of solid surrounding probe (cm^2/sec)

$\alpha_p = K_p/\rho_p c_p$ = thermal diffusivity of probe (cm^2/sec)

$C = \rho c$ = volumetric heat capacity of solid surrounding probe (cal/cm^3-°C)

$C_p = \rho_p c_p$ = volumetric heat capacity of probe (cal/cm^3-°C)

J_i = Bessel function of the first kind of order i

Y_i = Bessel function of the second kind of order i

I_i = modified Bessel function of the first kind of order i

K_i = modified Bessel function of the second kind of order i

$\gamma = .5772\ldots = \lim_{n\to\infty} \{1 + (1/2) + (1/3) + \ldots + (1/n) - \ln(n)\}$

= Euler's constant

$-Ei(-x) = \gamma - \ln(x) + x/(1\cdot1!) - x^2/(2\cdot2!) + x^3/(3\cdot3!) - \ldots$

R = radius of cylinder of material which surrounds the cylindrical probe (cm)

O(x) = remainder term of which the highest order term is of the order of x.

2.1.2 Definition of thermal conductivity

Consider a solid bounded by two parallel planes held at constant temperatures T_1 and T_2 ($T_2 > T_1$) respectively, and located a distance d apart. Imagine a cylindrical surface consisting of a set of parallel lines such that each generating line, and therefore the cylindrical surface, is perpendicular to the planes bounding the solid. Suppose the cylinder is restricted such that its intersection with the boundary planes encloses an area S (see Figure 2-1). Note that because of the definition of a cylinder [Protter and Morrey 1964, p. 555], the area S may be of any shape or size. After sufficient time has elapsed so that a steady state temperature distribution has been reached, the thermal conductivity K is defined by [Carslaw and

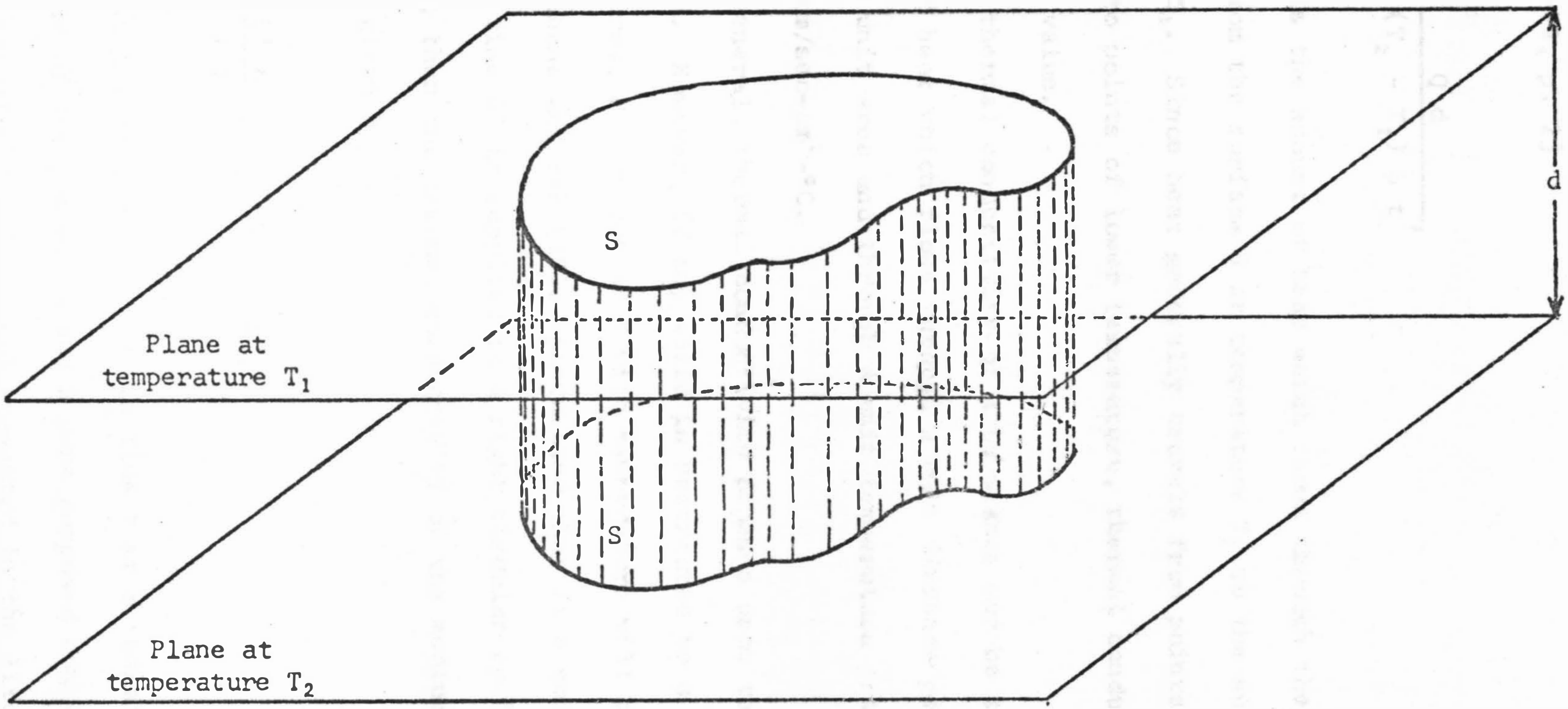

Figure 2-1. Two Isothermal Surfaces S Contained in Two Parallel Planes and Bounded by a Cylindrical Surface.

Jaeger 1959, p. 2]

$$K = \frac{Q\ d}{(T_2 - T_1)\ S\ t}, \tag{2-1}$$

where Q is the amount of heat which flows through the distance d in time t from the surface S at temperature T_2 to the surface S at temperature T_1. Since heat generally travels from points of higher temperature to points of lower temperature, thermal conductivity has a positive value.

The thermal conductivity of a substance may be thought of as the amount of heat which flows through a unit distance per unit time across a unit area and through a unit temperature gradient. Units are cal-cm/sec-cm^2-°C.

In general, thermal conductivity depends upon the temperature of the solid. However, if the solid is restricted to a limited range of temperatures, any change in K with temperature will be negligible.

Stalhane and Pyk [1931] postulated that if a constant rate of heat per unit time Q' is supplied to a right circular cylinder probe of length ℓ, then the thermal conductivity of the medium surrounding the probe is given by

$$K = \frac{Q'\ A}{\ell\ T}\ [\log(t/r^2) + B], \tag{2-2}$$

where T is the temperature rise in time t at a radial distance r from the center of the probe. A and B were supposed constants. Mathematical proof of equation (2-2) was not presented in the literature for many years (see §2.2).

2.1.3 The concept of heat flux

The rate at which heat flows across a surface at a point P per unit area per unit time is called the flux of heat at point P across the surface.

The flow of heat through the cylinder of §2.1.2 per unit area per unit time is given by

$$\frac{Q}{S\ t} = \frac{K\ (T_2 - T_1)}{d}, \qquad (2\text{-}3)$$

from equation (2-1). The heat flux at a point on one of the boundary surfaces is found by taking the limit of (2-3) as the distance d approaches the elemental distance ∂n and temperature difference $T_2 - T_1$ approaches ∂T. The flux of heat across a surface is therefore given by

$$f = -K \frac{\partial T}{\partial n}, \qquad (2\text{-}4)$$

where $\partial/\partial n$ indicates differentiation along a direction normal to the surface. The negative sign indicates that the heat flux flows in the direction of negative temperature gradient, i.e. from points of higher temperature to points of lower temperature.

In the above discussion the surface across which the heat flux of (2-4) flows was assumed to be isothermal. In other words all points on the surface are at the same temperature. The equation holds, however, for flux across any surface [Carslaw and Jaeger 1959, p. 7]. In the later sections of this chapter only flux across an isothermal surface is considered.

2.1.4 Equation of heat conduction

Since the problems to be considered in later sections of this chapter are best solved in cylindrical coordinates, the differential equation of heat conduction will be derived in cylindrical coordinates. This derivation follows closely that given for rectangular coordinates by Carslaw and Jaeger [1959, pp. 8-9].

An elemental volume in a solid within which no heat is generated is shown in Figure 2-2. The heat flux f across any surface through the point $P(r_0,\theta_0,z_0)$ at the center of the volume is a continuous function of r, θ, z, and t. The same is true for the temperature T of the point P.

2.1.4.1 Gain of heat in the r-direction

Note that the elemental volume of Figure 2-2 has linear dimensions 2dr and 2dz and angular dimension 2dθ. Consider a right circular cylinder with axis at the z-axis and radius r_0. The area of the cylinder cut off by the surfaces $\theta = \theta_0 \pm d\theta$ and $z = z_0 \pm dz$ is

$$S_r = (2r_0 d\theta)\,(2dz) = 4r_0 d\theta dz. \qquad (2\text{-}5)$$

The rate at which heat flows across the section S_r is just the multiple of the area S_r and the heat flux across S_r, or

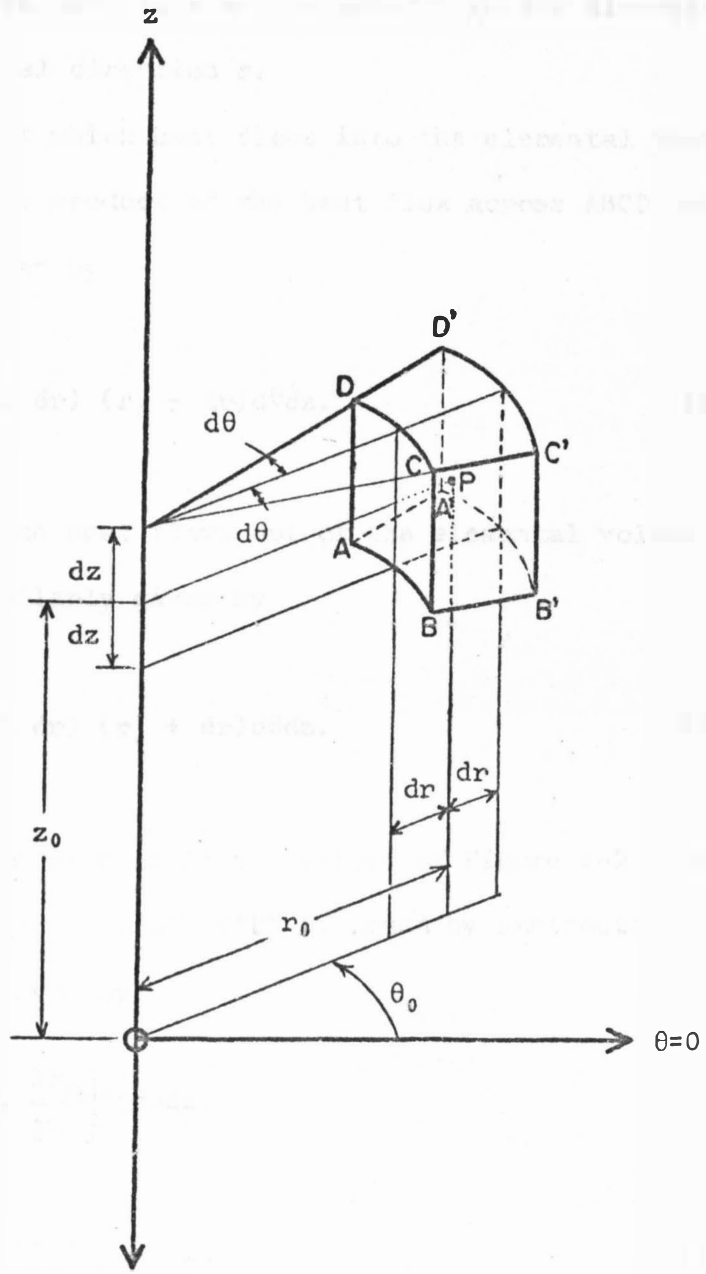

Figure 2-2. Elemental Volume in the Cylindrical Coordinate System

$$4f_r r_0 d\theta dz, \tag{2-6}$$

where f_r denotes heat flux at the point P in the direction of increasing radial direction r.

The rate at which heat flows into the elemental volume across face ABCD is the product of the heat flux across ABCD and the area of ABCD and is given by

$$4(f_r - \frac{\partial f_r}{\partial r} dr)\,(r_0 - dr)d\theta dz. \tag{2-7}$$

The rate at which heat flows out of the elemental volume across face A'B'C'D' is similarly given by

$$4(f_r + \frac{\partial f_r}{\partial r} dr)\,(r_0 + dr)d\theta dz. \tag{2-8}$$

The rate of gain of heat of the volume of Figure 2-2 from flow across the two faces ABCD and A'B'C'D' is found by subtracting (2-8) from (2-7), and is given by

$$-8\left[f_r + r_0 \frac{\partial f_r}{\partial r}\right]drd\theta dz,$$

$$\text{or} \quad -\left[8 \frac{\partial(rf_r)}{\partial r} drd\theta dz\right]_{r=r_0}. \tag{2-9}$$

2.1.4.2 <u>Gain of heat in the θ-direction</u>

Consider a plane containing both the z-axis and the point P of

Figure 2-2. Define the area S_θ as the section of the plane bounded by surfaces $r = r_0 \pm dr$ and $z = z_0 \pm dz$. The area S_θ is given by

$$S_\theta = (2dr)(2dz) = 4drdz. \qquad (2\text{-}10)$$

The rate at which heat flows across the surface S_θ at the point P is given by

$$4f_\theta drdz. \qquad (2\text{-}11)$$

The elemental volume gains heat across surface BB'C'C at the rate given by

$$4(f_\theta - \frac{\partial f_\theta}{\partial \theta} d\theta)drdz \qquad (2\text{-}12)$$

and loses heat across the surface AA'D'D at the rate given by

$$4(f_\theta + \frac{\partial f_\theta}{\partial \theta} d\theta)drdz. \qquad (2\text{-}13)$$

The total rate of gain of heat across these two surfaces is therefore

$$-8 \frac{\partial f_\theta}{\partial \theta} drd\theta dz. \qquad (2\text{-}14)$$

2.1.4.3 Gain of heat in the z-direction

Consider the area S_z on a plane through point P perpendicular to the z-axis bounded by surfaces $\theta = \theta_0 \pm d\theta$ and $r = r_0 \pm dr$ (Figure 2-2). The area S_z is given by

$$S_z = (2r_0 d\theta)\,(2dr) = 4r_0 dr d\theta. \tag{2-15}$$

The rate at which heat flows upward (in the direction of increasing z) across area S_z is given by

$$4f_z r_0 dr d\theta, \tag{2-16}$$

where f_z denotes heat flux at the point P across surface S_z.

The rate at which heat flows into the elemental volume across face AA'B'B is given by

$$4\left(f_z - \frac{\partial f_z}{\partial z}\, dz\right) r_0 dr d\theta, \tag{2-17}$$

and the rate at which the volume loses heat across surface CC'D'D is given by

$$4\left(f_z + \frac{\partial f_z}{\partial z}\, dz\right) r_0 dr d\theta. \tag{2-18}$$

The total rate of gain of heat across these two surfaces is

$$-8\,\frac{\partial f_z}{\partial z}\, r_0 dr d\theta dz. \tag{2-19}$$

2.1.4.4 Differential equation of heat conductivity

The total rate of gain of heat of the elemental volume in Figure 2-2 is the sum of the heat gained across all of its surfaces. From the results of §2.1.4.1, §2.1.4.2, and §2.1.4.3, given by expressions (2-9), (2-14), and (2-19), respectively, the total rate is given by

$$\frac{\partial Q}{\partial t} = -8\ dr d\theta dz \left[\frac{\partial(rf_r)}{\partial r} + \frac{\partial f_\theta}{\partial \theta} + r\frac{\partial f_z}{\partial z}\right]. \tag{2-20}$$

The subscript has been dropped from the variable r to emphasize that this expression holds for an elemental volume about any point $P(r,\theta,z)$, where the heat flux at the point P is completely described by its components f_r, f_θ, and f_z.

In §2.1.3 the heat flux f across a surface S was given by equation (2-4) as

$$f = -K\frac{\partial T}{\partial n}, \tag{2-4}$$

where K is thermal conductivity and $\partial/\partial n$ denotes differentiation in a direction normal to the surface S. The components of total heat flux are therefore given by

$$f_r = -K\frac{\partial T}{\partial r}, \tag{2-21a}$$

$$f_\theta = -\frac{K}{r}\frac{\partial T}{\partial \theta} \tag{2-21b}$$

$$f_z = -K\frac{\partial T}{\partial z} \tag{2-21c}$$

Substitution of equations (2-21) into expression (2-20) yields an expression for the total rate of gain of heat of an elemental volume about a point $P(r,\theta,z)$ given by

$$\frac{\partial Q}{\partial t} = 8\ K\ drd\theta dz \left\{\frac{\partial}{\partial r}\left[r\ \frac{\partial T}{\partial r}\right] + \frac{1}{r}\frac{\partial^2 T}{\partial \theta^2} + r\ \frac{\partial^2 T}{\partial z^2}\right\}$$

$$= 8\ K\ drd\theta dz \left[\frac{\partial T}{\partial r} + r\ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial^2 T}{\partial \theta^2} + r\ \frac{\partial^2 T}{\partial z^2}\right] \qquad (2\text{-}22)$$

Now consider a physical property of solids known as specific heat c. The specific heat of a substance at temperature T is defined [Carslaw and Jaeger 1959, p. 9] as the quantity of heat necessary to raise the temperature of a unit quantity of mass of the substance by the elemental temperature ∂T. If an elemental volume dV of a substance has specific heat c and density ρ, then the rate of gain of heat of the volume is given by

$$\frac{\partial Q}{\partial t} = \rho c\ dV\ \frac{\partial T}{\partial t},$$

or, in cylindrical coordinates, as in Figure 2-2,

$$\frac{\partial Q}{\partial t} = 8\rho c\ \ r\ drd\theta dz\ \frac{\partial T}{\partial t} \qquad (2\text{-}23)$$

Equating (2-22) with (2-23) yields

$$\frac{K}{\rho c}\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right] = \frac{\partial T}{\partial t}. \qquad (2\text{-}24)$$

The ratio $K/\rho c$ is called the thermal diffusivity, and shall hereafter be denoted by the symbol α. As with thermal conductivity K, the thermal diffusivity α of an isotropic, homogeneous solid may be considered constant over limited temperature ranges. In terms of thermal diffusivity the partial differential equation describing heat conduction in an infinite, homogeneous, isotropic solid in which no heat sources are present is given by

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right]. \tag{2-25}$$

2.2 Infinite line source of constant heat within an infinite medium

The above equation (2-25) describes the temperature T at any point $P(r,\theta,z)$. Since there are no heat sources present, the temperature will eventually become constant throughout the entire solid and no flow of heat will occur ($f_r = f_\theta = f_z = 0$). If the steady state temperature, say $T = 0$, has been reached in an infinite solid, and if at a later time, say $t = 0$, an infinitely long line along the z-axis is allowed to produce a constant rate of heat q per unit length per unit time, then the total flux f is given by f_r and equation (2-25) reduces to [Buettner 1955a; de Vries and Peck 1958a; van der Held and van Drunen 1949]

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right], \quad r > 0, \quad t > 0. \tag{2-26}$$

The assumption that the steady state temperature T = 0 has been reached throughout the solid before time t = 0 can be stated as

$$\lim_{t \to 0+}(T) = 0. \tag{2-27}$$

Consider a right circular cylinder of radius r about the z-axis. The flux of heat across an elemental area on the surface of the cylinder is given by equation (2-21a) as

$$f_r = -K \frac{\partial T}{\partial r}. \tag{2-21a}$$

The assumption that the line along the z-axis produces a constant rate of heat q per unit length per unit time is equivalent to stating that the product of the heat flux f_r through the surface of the cylinder and the circumference of the cylinder is a constant q, as the radius of the cylinder approaches zero. In mathematical form this becomes

$$\lim_{r\to 0+}\left[-2\pi r\, K \frac{\partial T}{\partial r}\right] = q,$$

$$\text{or } \lim_{r\to 0+} r \frac{\partial T}{\partial r} = -\frac{q}{2\pi K}, \quad t > 0. \tag{2-28}$$

Another necessary boundary condition is that points far away from the z-axis are not affected by the heat from the line source, or

$$\lim_{r\to +\infty}(T) = 0, \quad t > 0. \tag{2-29}$$

The solution to equations (2-26) through (2-29) was given in the literature as early as 1949 [van der Held and van Drunen] as

$$T = -(q/4\pi K)\, \mathrm{Ei}(-r^2/4\alpha t), \tag{2-30}$$

which reduces to[1]

$$T = (q/4\pi K)\ [\ln(4\alpha t/r^2) - \gamma + O(1/t)] \qquad (2\text{-}31)$$

for values of time t much greater than $r^2/4\alpha$.

Note the similarity between solution (2-31) and that of Stalhane and Pyk [1931] given as equation (2-2) in §2.1.2. The constant A in (2-2) is given by

$$A = \frac{\ln(10)}{4\pi},$$

and B of (2-2) is given by

$$B = \frac{\ln(4\alpha) - \gamma}{\ln(10)}.$$

Note that B is not a general constant but depends upon the thermal diffusivity α.

Van der Held and van Drunen [1949] avoided the dependence upon material type by observing the temperature T at two different times and writing

$$(T_2 - T_1) = (q/4\pi K)\ \ln(t_2/t_1), \qquad (2\text{-}32)$$

where each T_i is the temperature at time t_i.

---

[1] A derivation of solutions (2-30) and (2-31) is given in Appendix I.

2.3 Method of measuring thermal conductivity

From the above discussion a method of determining thermal conductivity becomes apparent [van der Held and van Drunen 1949]. The method is independent of the thermal properties of the material but requires the use of an infinite line source of heat, approximated in practice by a cylindrical probe containing an electrical resistance per unit length to which a constant current is applied. The power input to the probe can be measured and the temperature rise can be observed and plotted against time. The plot approaches a straight line on semilog paper for large values of t. The value of the slope of the line, $q/4\pi K$, can be graphically determined. Thus, a value of thermal conductivity is obtained.

2.4 Infinite medium bounded internally by an infinitely long cylindrical[2] heat source

The accuracy of the method of §2.3 depends primarily on how closely an infinite line source of heat can be practically approximated. A mathematical expression for the temperature rise of points a distance r away from the center of the cylindrical probe, for r greater than the

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[2] The word "cylindrical" here means "having the shape of a right circular cylinder." This definition remains in effect throughout the remainder of this chapter.

radius of the probe b, has been published in the literature [van der Held, Hardebol, and Kalshoven 1953]. A derivation of the mathematical expression for the temperature rise is given in Appendix II. As shown in Appendix II, the solution reduces to equation (2-32) for large values of time t, and the method described in §2.3 is valid.

2.5 Contact-resistance at the boundary between the infinitely long cylindrical probe and the infinite medium

It has been pointed out [Blackwell 1954] that the above theoretical basis for the cylindrical probe method of measuring thermal conductivity is insufficient. Thermal contact-resistance between the surface of the probe and the external medium has heretofore been assumed negligible. In other words, the thermal conductivity of the boundary between probe and medium has been assumed infinite. The temperature of the probe at the boundary has therefore been tacitly assumed to be equal to the temperature of the medium at the boundary.

If one now adds the assumption of a temperature difference at the boundary and a finite thermal conductivity per unit length H across the boundary to the assumptions of the above theory, the equations to be solved, (2-26) through (2-29), become [Blackwell 1954]

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right], \quad r > b, \quad t > 0; \qquad (2\text{-}33a)$$

$$T = T_p = 0, \quad t = 0; \qquad (2\text{-}33b)$$

$$-K \frac{\partial T}{\partial r} = H\,(T_p - T), \quad r = b,\ t > 0; \tag{2-33c}$$

$$-2\pi b\ K \frac{\partial T}{\partial r} = q - M_p\ c_p \frac{\partial T_p}{\partial t}, \quad r = b, \quad t > 0; \tag{2-33d}$$

and $\lim\limits_{r\to+\infty}(T) = 0, \quad t > 0.$ (2-33e)

The solution for probe temperature is given by [Blackwell 1954]

$$T_p = (q/4\pi K)\ [\ln(4\alpha t/b^2) - \gamma + 2K/bH$$

$$+ (b^2/2\alpha t)\ \{\ln(4\alpha t/b^2) - \gamma + 1 - (C_p/C)\ (\ln[4\alpha t/b^2] - \gamma + 2K/bH)\}$$

$$+ O(1/t^2)], \tag{2-34}$$

which is valid for large values of time t. The main difference between (2-34) and earlier solutions (see (2-31)) is the inclusion of terms of order $1/t$ and $\ln(t)/t$. Note that as $H \to \infty$ and $C_p \to C$, terms of order $\ln(t)/t$ vanish and equation (2-34) reduces to equation (2-31).

For finite H, $C_p \neq C$, and very large values of t, even the terms of order $\ln(t)/t$ become negligible and (2-34) reduces to

$$T_p = (q/4\pi K)\ [\ln(4\alpha t/b^2) - \gamma + 2K/bH]. \tag{2-35}$$

However, a larger time t is required for $O[\ln(t)/t]$ in (2-34) to become negligible than is required for $O(1/t)$ in earlier solutions to vanish. Equation (2-35) is similar to the solution of §2.2 given by (2-31) except for the constant term involving contact-resistance. However, if the partial derivative of T_p is taken with respect to $\ln(t)$, one obtains the result

$$\frac{\partial T_p}{\partial[\ln(t)]} = \frac{q}{4\pi K}. \qquad (2\text{-}36)$$

Since T_p therefore varies linearly with $\ln(t)$ for large values of time t, equation (2-36) may be rewritten as (2-32) of §2.2, or

$$\frac{T_{p2} - T_{p1}}{\ln(t_2/t_1)} = \frac{q}{4\pi K}, \qquad (2\text{-}32)$$

where T_{pi} is the temperature at time t_i. Therefore, the method of §2.3 still applies for large values of time.

2.6 Infinitely long cylindrical heat source having finite thermal conductivity

None of the above solutions depend upon the thermal conductivity of the probe, K_p. In fact, it has been a tacit assumption that the probe conductivity is infinite [Blackwell 1954; Jaeger 1956]. Therefore, the temperature at any point of the probe has been assumed to be the same as that at any other point of the probe at a given instant.

If one now assumes that the probe is not a perfect conductor, then the solution for large values of time becomes [Blackwell 1954]

$$\begin{aligned} T_{p,a} = (q/4\pi K)\{&\ln(4\alpha t/b^2) - \gamma + 2K/bH \\ &+ (b^2/2\alpha t)\,[\ln(4\alpha t/b^2) - \gamma + 1 - (C_p/C)\,(\ln[4\alpha t/b^2] - \gamma + 2K/bH)] \\ &- (1/4\alpha_p t)\,[b^2 - a^2 + 2a^2\ln(b/a)] + O(1/t^2)\}, \end{aligned} \qquad (2\text{-}37)$$

where $T_{p,a}$ is the temperature along the inner surface r = a of a hollow cylindrical probe. The solution (2-37) differs from solution (2-34) in only one term of order 1/t, the value of which becomes negligible for thin-walled probes where (b-a) << b [Blackwell, 1954]. For very large values of time t, equation (2-37) reduces to (2-35). The partial derivative of $T_{p,a}$ with respect to ln(t) is given by (2-36) as a constant inversely proportional to K. The method of §2.3 therefore still applies.

2.7 Infinite medium bounded internally by a cylindrical heat source of finite length

All of the above discussion assumes an infinitely long cylindrical probe source of heat contained within an infinite solid. The implication is that all heat flow has been assumed to be in a radial direction away from the probe and no heat flow occurs in the direction of the probe axis.

Blackwell [1956] discusses the error involved in making this assumption. For probes of finite length he writes

$$\frac{\partial T_{p,a,0}}{\partial[\ln(t)]} = \frac{q}{4\pi K} G \quad t \gg b^2/\alpha, \; t \gg b^2/\alpha_p, \qquad (2\text{-}38)$$

where $G = \operatorname{erf}(x) - \dfrac{2\ D\ S\ x^3 \exp(-x^2)}{\sqrt{\pi}}$

$$+ x^4 \frac{D^2}{\sqrt{\pi}} \left[3S^2 - \frac{\pi^2}{2}\right]\left[\frac{1}{2} + \frac{x}{2} - x^2\right] \exp(-x^2);$$

$$x = \frac{\ell}{4\sqrt{\alpha t}},$$

$$D = 4 \frac{K_p}{K} \left[1 - \frac{\alpha}{\alpha_p}\right] \left[\frac{b^2 - a^2}{\ell^2}\right],$$

and $\quad S = \ln(4\alpha t/b^2) - \gamma + 2K/bH.$

As $\ell \to \infty$, $G \to 1$ and equation (2-36) is the result. The first term of G does not depend upon thermal properties of the probe and may be thought of as the contribution to axial flow due to finite probe length alone. The remaining terms include the axial heat flow due to the finite probe radius b and contact-resistance 1/H; they approach zero as $\ell \to \infty$.

For finite probe length ℓ the value of G decreases as t increases. The error due to values of G less than unity may be expressed as the difference between unity and the actual value of G, i.e. $\Delta G = 1 - G$. This error is less than 0.0004 provided that [Blackwell, 1956]

$$t < \frac{\ell^2}{64\,\alpha}. \tag{2-39}$$

2.7.1 Numerical example

For a probe of length $\ell = 10$ cm, radius $b = 0.05$ cm, and high diffusivity ($\alpha_p > \alpha$), placed in an infinite medium of diffusivity $\alpha = 0.005\ cm^2/sec$, the expression (2-36) is valid for values of time $t \gg b^2/\alpha = 0.5$ sec and $t < \ell^2/64\alpha \simeq 310$ sec.

2.8 Cylindrical heat source within a finite medium

The effects of finite properties of the cylindrical probe have been investigated in detail, but the theoretical solutions are strictly valid only for the case of an infinite solid bounded internally by the cylindrical probe. The effects of using solution (2-36) to find thermal conductivity K of materials of finite dimensions was studied by Golovanov [1969].

If the surface of a small sample of material bounded internally by a heat source is kept at a constant temperature, and if the material is allowed to reach thermal equilibrium before the heat source is turned on, then it is intuitively obvious that the temperature rise within the small sample will be less than that in an infinite sample. The value of thermal conductivity calculated by the use of equation (2-36) will then be too large. On the other hand, if the surface of the sample is well insulated, the temperature rise will be greater than that in an infinite sample. The calculated K will therefore be too small. The magnitude of error depends upon the thermal properties of the material and the length of time of measurement.

The theory of Golovanov applies for an infinitely long cylinder of radius R containing a material of thermal constants K and α, bounded internally by an infinite line source of heat. The temperature rise T' within the cylinder with constant temperature $T_0 = 0$ at its surface is given by [Golovanov 1969]

$$T' = \frac{q}{4\pi K}\left\{2\ln\left[\frac{R}{r}\right] - 4\sum_{n=1}^{\infty}\left[\frac{J_0\left[\frac{u_n r}{R}\right]\exp\left[-\frac{u_n^2 \alpha t}{R^2}\right]}{u_n^2 J_1^2(u_n)}\right]\right\} \quad (2\text{-}40)$$

where u_n = roots of $J_0(u_n) = 0$. The temperature rise T'' within the cylinder with perfectly insulated surfaces is given by

$$T'' = \frac{q}{4\pi K}\left\{2\ln\left[\frac{R}{r}\right] - \left[\frac{r}{R}\right]^2 + \left[\frac{4\alpha t}{R^2}\right] - \frac{3}{2}\right.$$

$$\left. - 4\sum_{n=1}^{\infty}\left[\frac{J_0\left[\frac{v_n r}{R}\right]\exp\left[-\frac{v_n^2 \alpha t}{R^2}\right]}{v_n^2 J_0^2(v_n)}\right]\right\}, \qquad (2\text{-}41)$$

where v_n = roots of $J_1(v_n) = 0$.

With the aid of a digital computer, Golovanov tabulated values of $4\pi KT'/q$, $4\pi KT''/q$, and $4\pi KT/q$ (from equation (2-30)) for r = 0.05 cm, R = 1, 2, 3, 5, 7, and 10 cm, and values of αt ranging from 0 to 6 cm^2. Golovanov then computed the minimum radius R_{min} for which equation (2-36) can be used for determining thermal conductivity with an error less than 1%. The minimum radius R_{min} depends upon α and the time of measurement t and is given approximately by

$$R_{min} \simeq 1.9\sqrt{\alpha t} \quad \left(\text{for } \left|\frac{\Delta K}{K}\right| < 1\%\right). \qquad (2\text{-}42)$$

From the experimental data of Golovanov [1969], effects of finite probe characteristics do not affect the result (2-42) provided that

$$t >> b^2/\alpha, \qquad (2\text{-}43a)$$

$$t >> b^2/\alpha_p, \qquad (2\text{-}43b)$$

and $$t << \ell^2/\alpha. \qquad (2\text{-}43c)$$

2.9 Summary

The method of §2.3 for measuring thermal conductivity was first suggested by Stalhane and Pyk [1931] (see equation (2-2)). It was proven [van der Held and van Drunen 1949], as shown in §2.2, that the method is strictly valid for an infinite solid containing an infinite line source of heat. The latter sections of this chapter show that if precautions are taken as to the physical properties and dimensions of the cylindrical probe and the length of time of measurement, then the method of §2.3 remains valid. Many investigators have found good experimental success in using this method of measuring thermal conductivity [Mason and Kurtz 1952; Van Duin and de Vries 1954; Buettner 1955b; Bullard, Maxwell, and Revelle 1956; deVries and Peck 1958a; Von Herzen and Maxwell 1959; Corry, Dubois, and Vacquier 1968; Wierenga, Nielsen, and Hagan 1969]. The method does not work as well for measuring conductivity of liquids, however, because of convection of heat near the probe [van der Held and van Drunen 1949; van der Held, Hardebol, and Kalshoven 1953].

CHAPTER THREE

INSTRUMENT TO MEASURE THERMAL RESISTIVITY

3.1 Introduction

In §2.3 a method for measuring thermal conductivity was presented. The method consists of finding the slope of a plot of temperature versus log(time). The following sections describe in detail the components of a system which finds thermal resistivity (the reciprocal of thermal conductivity) without the need for the manual labor involved in obtaining data for a plot of temperature. The accuracy of the system is discussed in Chapter 4.

As was pointed out in Chapter 2, the change of temperature with respect to the logarithm of time becomes constant under appropriate conditions. The constant was given in equation (2-36) as $q/4\pi K$, where q is the heat supplied per unit time per unit length and K is thermal conductivity.

The system described in the following sections automatically applies a constant current to a resistive heater for a predetermined length of time. The temperature near the midpoint of the heater is measured at times t_i such that t_i/t_{i-1} is constant. The change in temperature during the interval of time from t_{i-1} to t_i is calculated. This value is multiplied by a system constant which depends upon the heater current and gain elements within the measuring system. The result, equal to thermal resistivity 1/K in decimeter-second-°C/calorie, is displayed until the next measurement at time t_{i+1}.

After a predetermined time τ, the current supply to the heater is automatically turned off. Measurements of temperature are now taken at values of time t_k such that $(t_k - \tau)/(t_{k-1} - \tau)$ is constant. The magnitude of the change in temperature during the interval of time from t_{k-1} to t_k is calculated. This value is multiplied by the same constant as is used during the heating period. The result is displayed until the next measurement at time t_{k+1}. After a total measurement time 2τ, the instrument cuts the power to most of its circuitry to save energy. For further measurements, the instrument must be reset by turning the power switch to OFF and then back to ON.

The system consists of several functional modules. Each is explained in detail in the following sections. The constant current source and the cylindrical probe are described in §3.2. The temperature sensing circuitry is discussed in §3.3. In §3.4 the method of keeping the ratio of successive measurement times t_i/t_{i-1} constant is explained. The analog signal proportional to temperature is digitized as described in §3.5. When a new temperature measurement is taken, the value of the measurement immediately preceding is stored in memory. The magnitude of the difference between the values of the successive measurements is found by circuitry described in §3.6. Display circuitry and power supplies are discussed in §3.7 and §3.8, respectively.

The logic symbol conventions of Dinsmore [1974] are used wherever possible in the figures of this chapter.

3.2 Constant heat source

The theory of Chapter 2 applies for cylindrical sources of heat. A sketch of the heat source used in the system for measuring thermal resistivity is shown in Figure 3-1. Nickel-chromium resistance wire (Brown and Sharpe wire gauge #33, 13.5 Ω/foot) was wound around an 8-inch length of quarter-inch copper tubing. Total resistance of the coil is 790 Ω.

The probe heater is supplied by the constant current source shown in Figure 3-2. When a logic 1 voltage level is applied to the HEATER ON input, current is supplied to the probe heater. The value of the constant current is determined by resistors R_1, R_3, and R_9, potentiometer R_2, and the supply voltage, and it can be varied from 7.8 mA to 14 mA. The combination of A_1, Q_1, R_7, and R_8 ensures that the voltage drop across R_9 (and therefore the heater current) is kept constant. The value of heat supplied by the probe per unit time per unit length is given by

$$q = \frac{k\ I^2\ R}{\ell}.$$

For R = 790 Ω, ℓ = 20 cm, and k = .239 cal/watt-sec the value of q can be adjusted from .57 to 1.9 mcal/cm-sec.

When a logic zero voltage level is applied to the HEATER ON input, the voltage at the output of the open-loop amplifier A_2 becomes approximately equal to its negative supply voltage. Transistor Q_1 is therefore cut off and a negligible amount of current flows through the

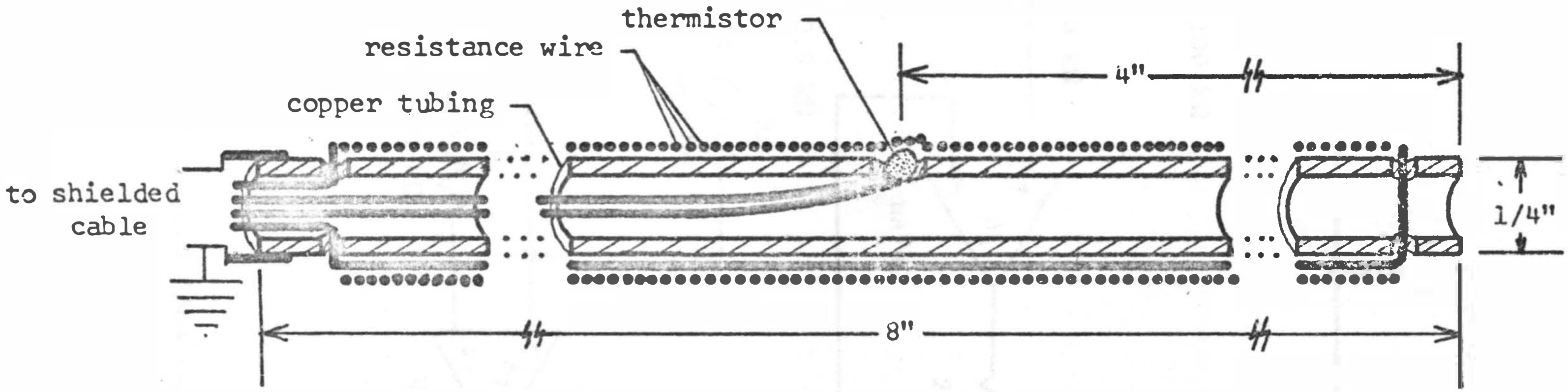

Figure 3-1. Cross Section of Cylindrical Probe

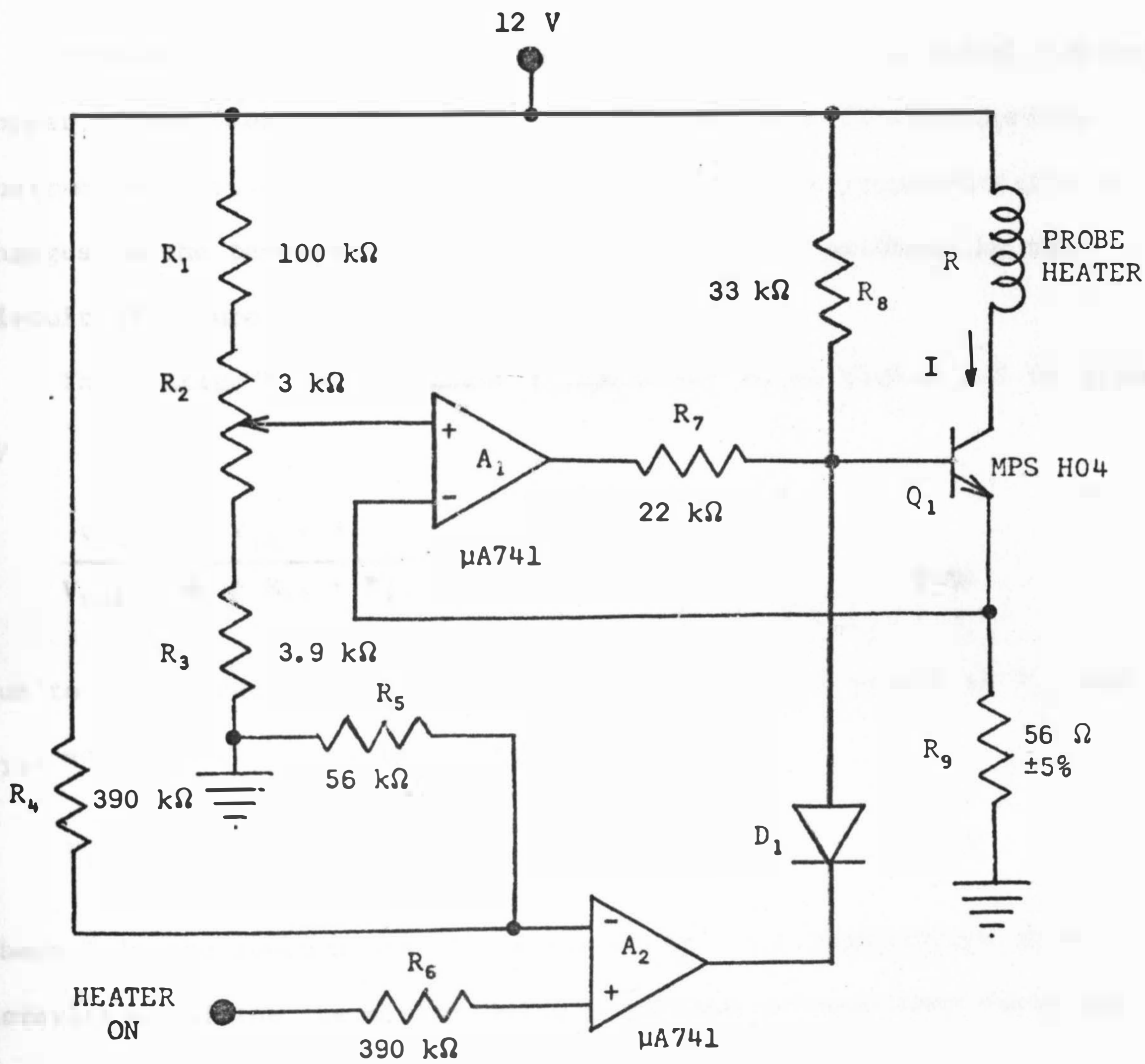

Figure 3-2. Constant Current Source

probe heater.

3.3 Temperature sensor

Mounted flush with the surface at the center of the 8-inch (20-cm) copper tubing (see Figure 3-1) is a 1-MΩ thermistor (Yellow Springs Instruments part #44015[1]). A voltage which changes proportionally to changes in the temperature of the thermistor R_T is produced by the circuit of Figure 3-3.

The voltage V_1 at the input to amplifier A_3 of Figure 3-3 is given by

$$\frac{V_1}{V_{ref}} = \frac{R_{10} + R_{11}}{R_T + R_{10} + R_{11}} \tag{3-1}$$

Due to the characteristics of thermistor R_T and the values of R_{10} and R_{11}, equation (3-1) may be approximated by

$$V_1 \,/\, V_{ref} \simeq 0.01250\ T + 0.2500, \tag{3-2}$$

where T is the temperature of thermistor R_T. The temperature characteristics of the thermistor and a comparison of equations (3-1) and (3-2) are tabulated in Table 3-1. Values of Δ% are found by taking the difference of each value in the third and fourth columns, then dividing by the value in the fourth column, finally multiplying times 100%.

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[1] Available from YSI Components Division, P.O. Box 279, Yellow Springs, Ohio 45387.

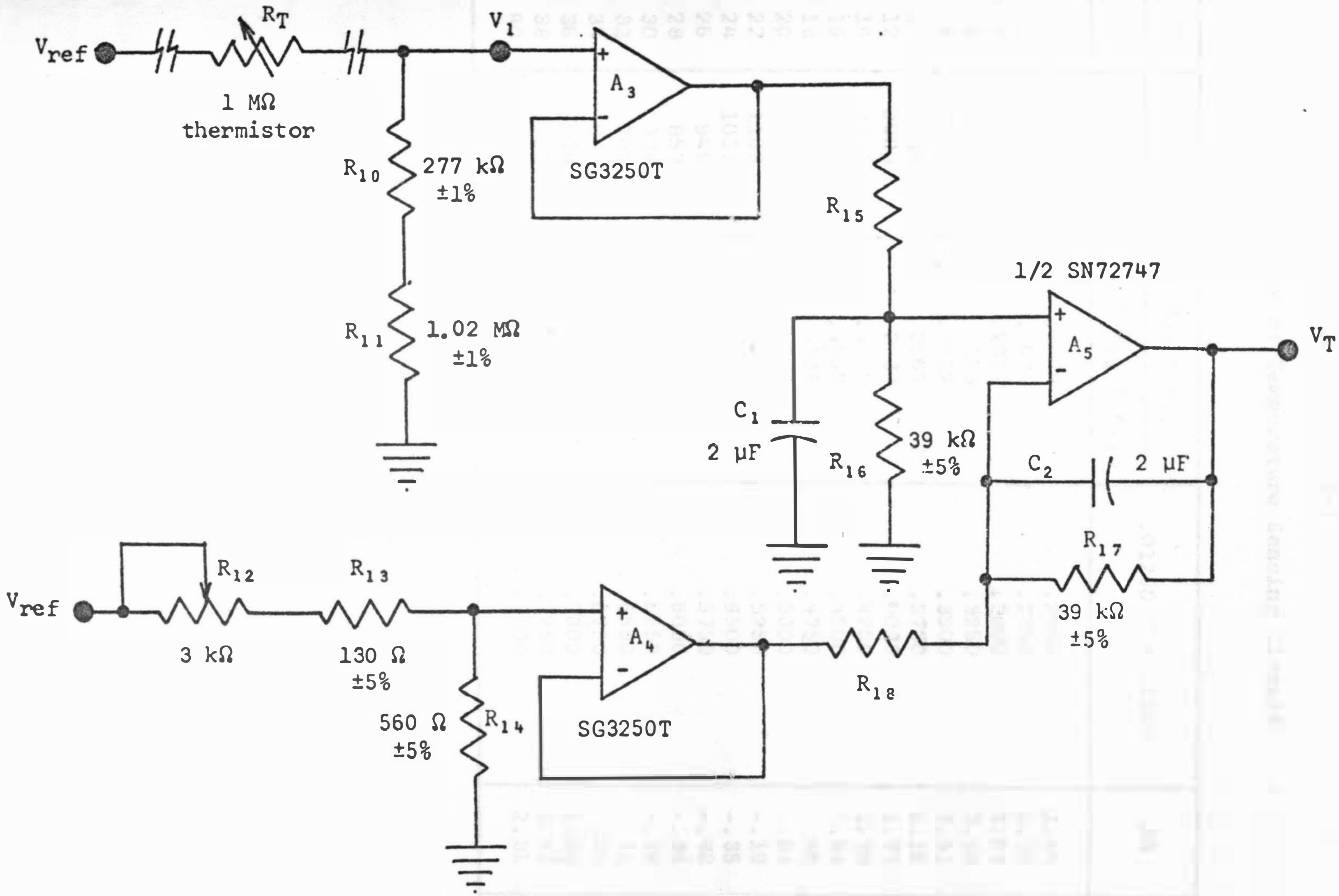

Figure 3-3. Temperature Sensing Amplifier

TABLE 3-1

Linearity of Temperature Sensing Circuit

T °C	R_T kΩ	$\frac{R_{10} + R_{11}}{R_T + R_{10} + R_{11}}$	.01250 T + .2500	Δ%
0	3966	.2464	.2500	1.44
2	3529	.2688	.2750	2.25
4	3144	.2921	.3000	2.63
6	2804	.3163	.3250	2.68
8	2504	.3412	.3500	2.51
10	2238	.3669	.3750	2.16
12	2003	.3930	.4000	1.75
14	1795	.4195	.4250	1.29
16	1610	.4462	.4500	.84
18	1446	.4728	.4750	.46
20	1299	.4996	.5000	.08
22	1169	.5260	.5250	-.19
24	1053	.5519	.5500	-.35
26	949.7	.5773	.5750	-.40
28	857.2	.6021	.6000	-.35
30	774.5	.6261	.6250	-.18
32	700.5	.6493	.6500	.11
34	634.1	.6716	.6750	.50
36	574.6	.6930	.7000	1.00
38	521.2	.7133	.7250	1.61
40	473.2	.7327	.7500	2.31

For any value of thermistor temperature between 0° C and 40° C, the output voltage V_T (see Figure 3-3) can be set to zero by adjusting R_{12}.

Since $R_{16} = R_{17}$ and $R_{15} = R_{18}$, the output voltage V_T is given by

$$V_T \simeq 0.0125\ V_{ref}\ (R_{16}/R_{15})\ (T - T_0),$$

where T_0 is determined by the setting of R_{12}.

3.4 Log(time) control

A low-frequency clock circuit is shown in Figure 3-4. As power is applied, the LF CLOCK output is low and capacitor C_3 charges through R_{20} and input resistors of I_1. The capacitor continues to charge until the LF CLOCK output goes high. Then the capacitor begins to discharge through R_{19} and R_{20} until the output returns to a low level. With C_3 = 600 μF, clock frequency is about 0.68 Hz.

A circuit which converts the simple chain of clock pulses from the low-frequency clock (where $t_i - t_{i-1}$ is constant) to a logarithmic chain of pulses (where $\log(t_i) - \log(t_{i-1})$ is constant) is shown in Figure 3-5. The symbol t_i represents the time at which the ith clock pulse occurs. A pulse is present at the LOG CLOCK output on the first, second, fourth, eighth, etc. input pulse. Therefore, $\log(t_i) - \log(t_{i-1}) = \log(t_i/t_{i-1}) = \log(2)$.

Circuit operation is understood by noting that odd-numbered flip-flops FF_1 through FF_{15} and FF_{16} form a nine-stage binary ripple counter. Even-numbered flip-flops FF_2 through FF_{14} are connected so that if one becomes SET, it remains SET until a low level is applied to the Clear

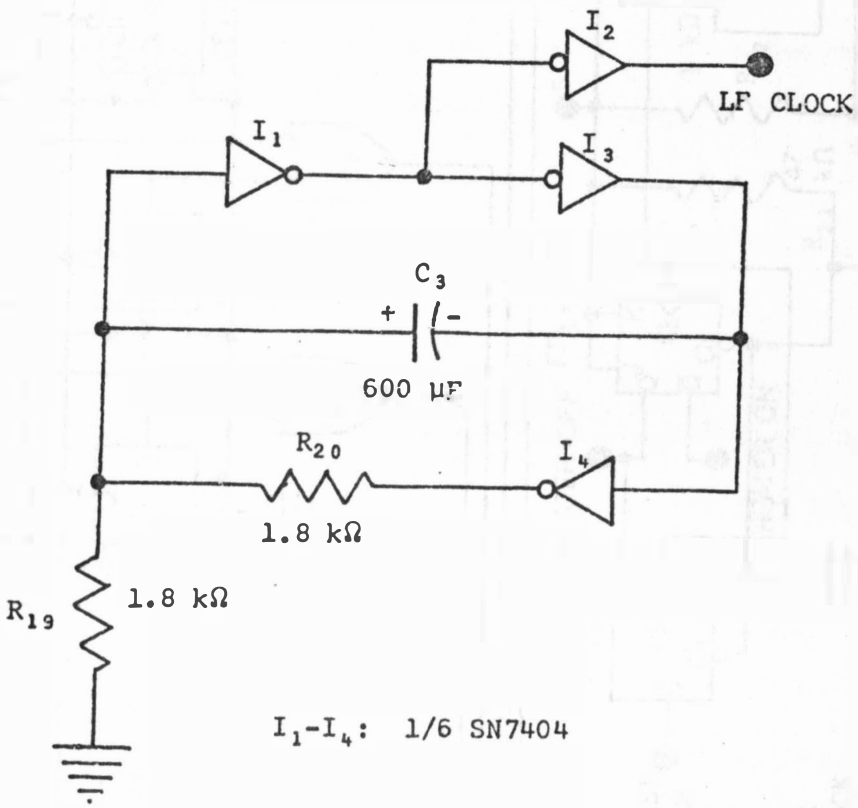

Figure 3-4. Low-frequency Clock

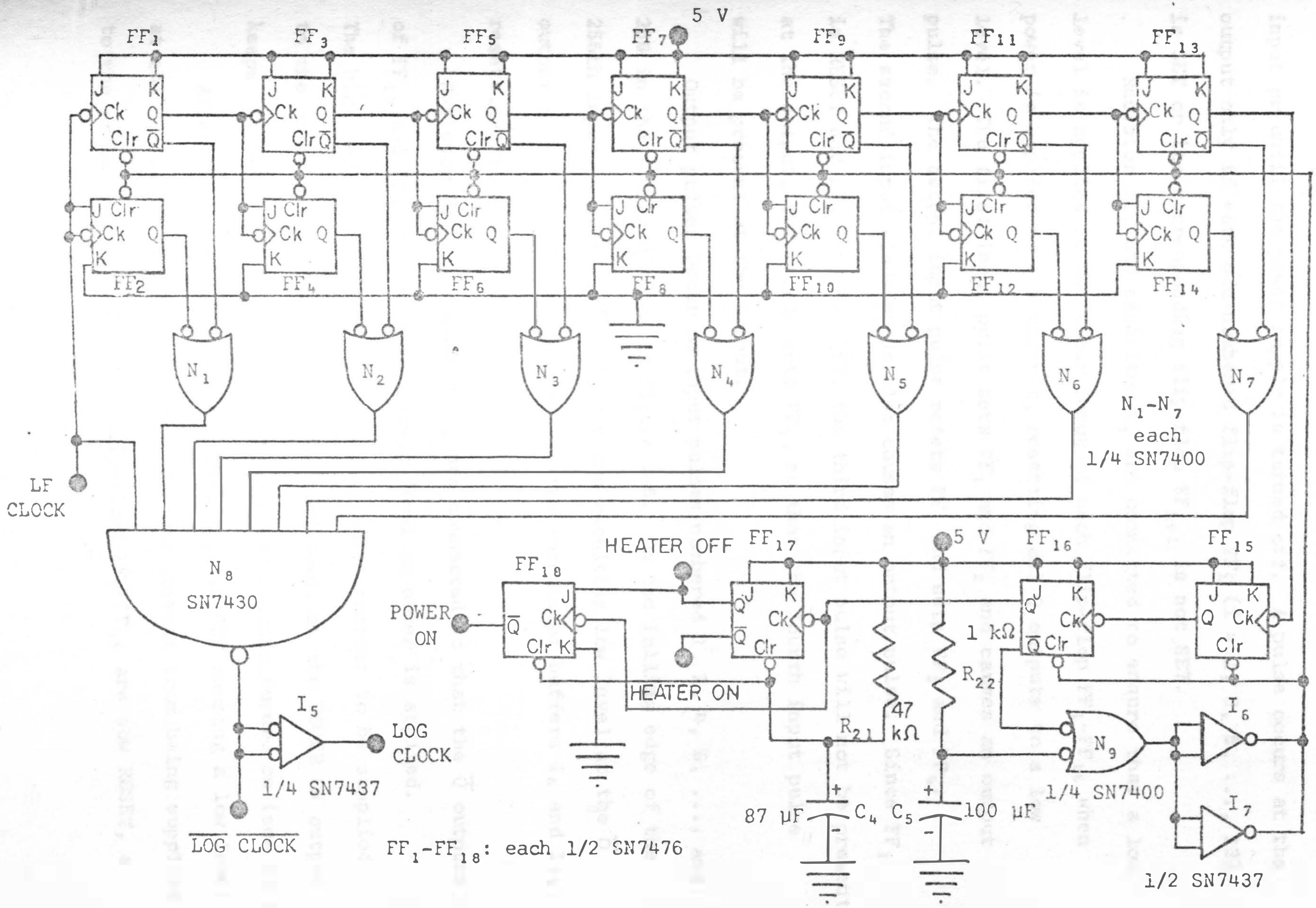

Figure 3-5. Log Clock

input or until the power supply is turned off. A pulse occurs at the output only if each odd-numbered flip-flop FF_i (i = 1, 3, 5, ..., 13) is SET or its corresponding flip-flop FF_{i+1} is not SET.

Resistor R_{22} and capacitor C_5 are connected to ensure that a low level is applied to the Clear input of each flip-flop FF_1-FF_{16} when power is applied to the circuit, resetting all Q outputs to a low level. The first input pulse sets FF_1 and FF_2 and causes an output pulse. The second input pulse resets FF_1 and sets FF_3 and FF_4. The second input clock pulse also causes an output pulse. Since FF_1 is RESET and FF_2 is still SET, the third input pulse will not be present at the output, but merely sets FF_1, so that the fourth input pulse will be present at the output.

Output pulses occur on input pulses numbered 1, 2, 4, 8, ..., and 256 in the timing diagram of Figure 3-6. On the falling edge of the 256th input pulse FF_{16} is SET, but the resulting low level at the $\overline{Q}$ output of FF_{16} causes a low level at the outputs of buffers I_6 and I_7, resetting flip-flops FF_1-FF_{16}.

Resistor R_{21} and capacitor C_4 are connected so that the $\overline{Q}$ outputs of FF_{17} and FF_{18} are at a high logic level as power is applied. The high level at the HEATER ON output causes current to be supplied to the probe heater (see §3.2). The high level at the POWER ON output keeps the 5-V supply for the TTL integrated circuits turned on (see §3.8).

After the 256th input clock pulse, FF_{17} is SET causing a low level at the HEATER ON output, preventing further current from being supplied to the probe heater. Since all flip-flops FF_1-FF_{16} are now RESET, a

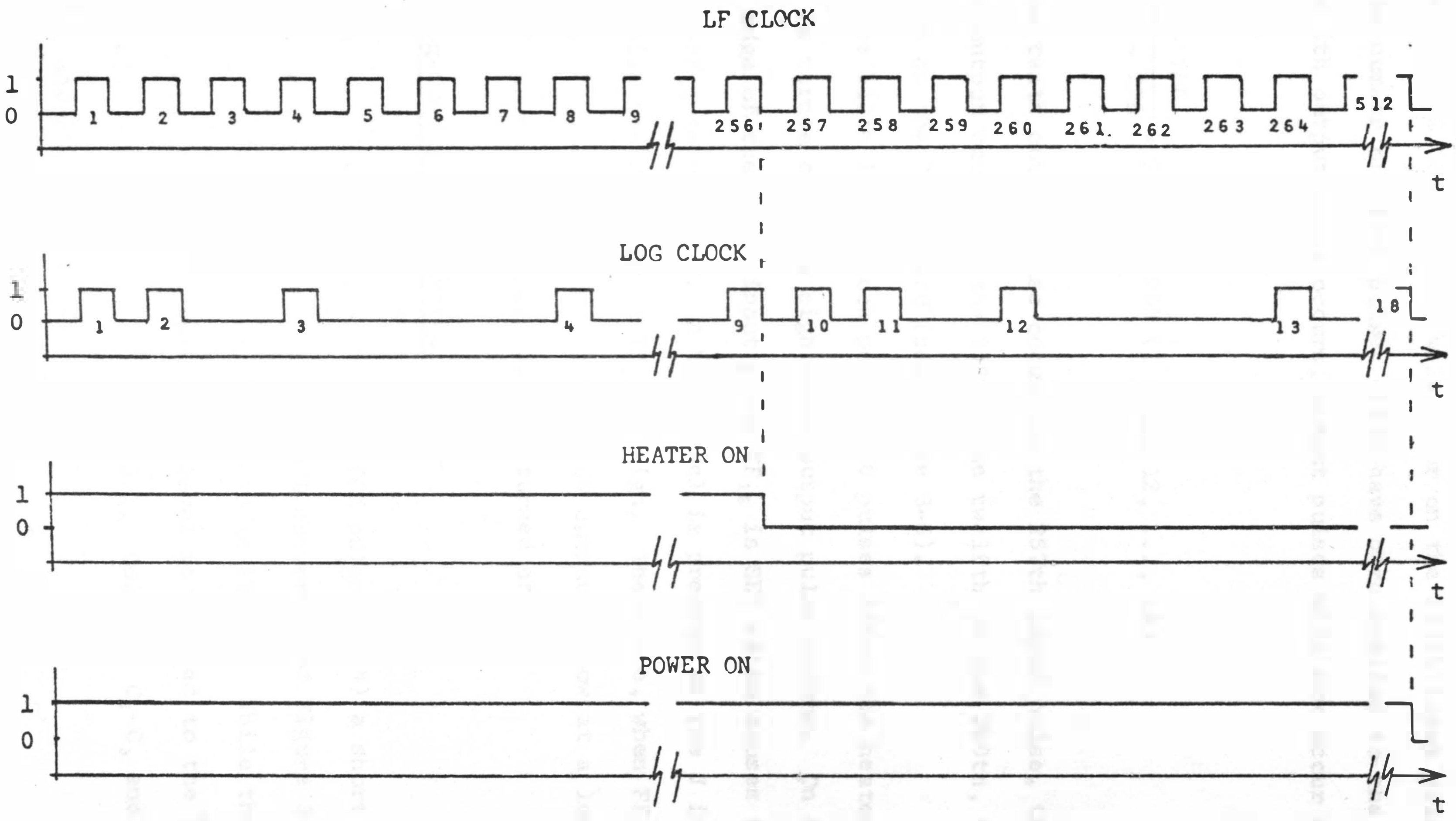

Figure 3-6. Timing Diagram for Circuitry of Figure 3-5

pulse at the LOG CLOCK output will occur on the 257th input pulse. If n_i is the number of clock pulses which have been applied to the input when the ith output pulse occurs, output pulses will now occur only when

$$\frac{n_i - 256}{n_{i-1} - 256} = 2,\ n_i > 257\ (i = 11, 12, \ldots, 18).$$

Since the tenth output pulse occurs on the 257th input pulse, the eleventh output occurs on the 258th, the twelfth on the 260th, the thirteenth on the 264th, etc (see Figure 3-6).

On the 512th input clock pulse (256 pulses after the heater current is turned off) the eighteenth output pulse occurs. On the falling edge of the 512th input pulse FF_{16} is SET which causes flip-flops FF_1-FF_{16} to be RESET. A high level is present at the J input of FF_{18} while the Q output of FF_{16} is high. Therefore, when FF_1-FF_{16} are RESET, FF_{18} is SET. But the POWER ON output is now at a low level and the TTL power supply (see §3.8) is turned off.

3.5 Analog-to-digital converter

On the falling edge of each LOG CLOCK pulse (§3.4) a short pulse is applied to the $RESET_1$ input of the A/D converter of Figure 3-7. Generation of the reset pulse is discussed in §3.5.1. While the $RESET_1$ input is at a high level, a low level is applied to the $\overline{RESET}$ input, as will also be explained in §3.5.1. Counters C_1-C_3 and flip-flops FF_{19} and FF_{20} are RESET.

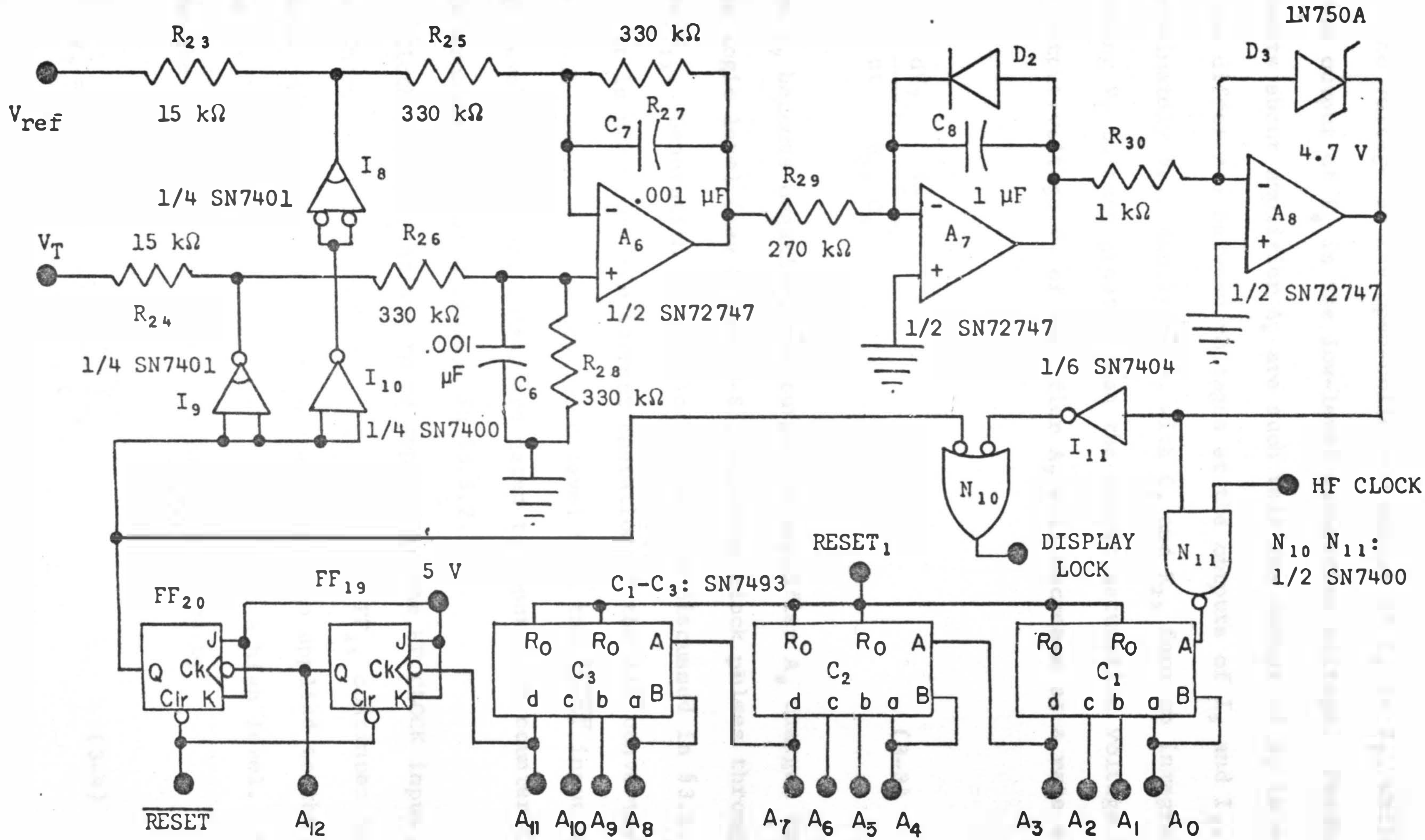

Figure 3-7. Schematic Diagram of A/D Converter

The voltage at the open-collector output of I_9 is V_T, while that at the output of I_8 is the low-level saturation voltage. Feedback elements about amplifier A_6 are such that the output of A_6 is equal to the difference between voltages at the outputs of I_9 and I_8, or approximately V_T. Amplifier A_7 with C_8 and R_{29} form an integrator. Assuming V_T is much greater than the output saturation voltage of I_8, the output voltage V_7 of amplifier A_7 will decrease at a rate equal to

$$-\frac{dV_7}{dt} \simeq \frac{V_T}{R_{29}\,C_8}. \qquad (3\text{-}3)$$

When V_7 becomes negative, the output of amplifier A_8 changes to a high logic level (see Figure 3-8), allowing clock pulses through NAND gate N_{11}. Generation of the clock pulses is discussed in §3.5.2.

It is necessary for correct operation of the A/D converter that the $RESET_1$ input be at a low logic level (and the $\overline{RESET}$ input at a high level) when clock pulses are gated to input A of counter C_1. This problem will be discussed in §3.5.2.

Clock pulses continue to be applied to the HF CLOCK input, and the binary counter formed by C_1-C_3, FF_{19}, and FF_{20} continues to count. After 2^{13} = 8192 clock pulses have been applied to the counter input, the Q output of FF_{20} changes to a high level. At this time the voltage V_7 at the output of A_7 is given by

$$V_7 \simeq \frac{dV_7}{dt}\,\frac{8192}{f_c} = -\frac{8192\,V_T}{f_c\,R_{29}\,C_8}, \qquad (3\text{-}4)$$

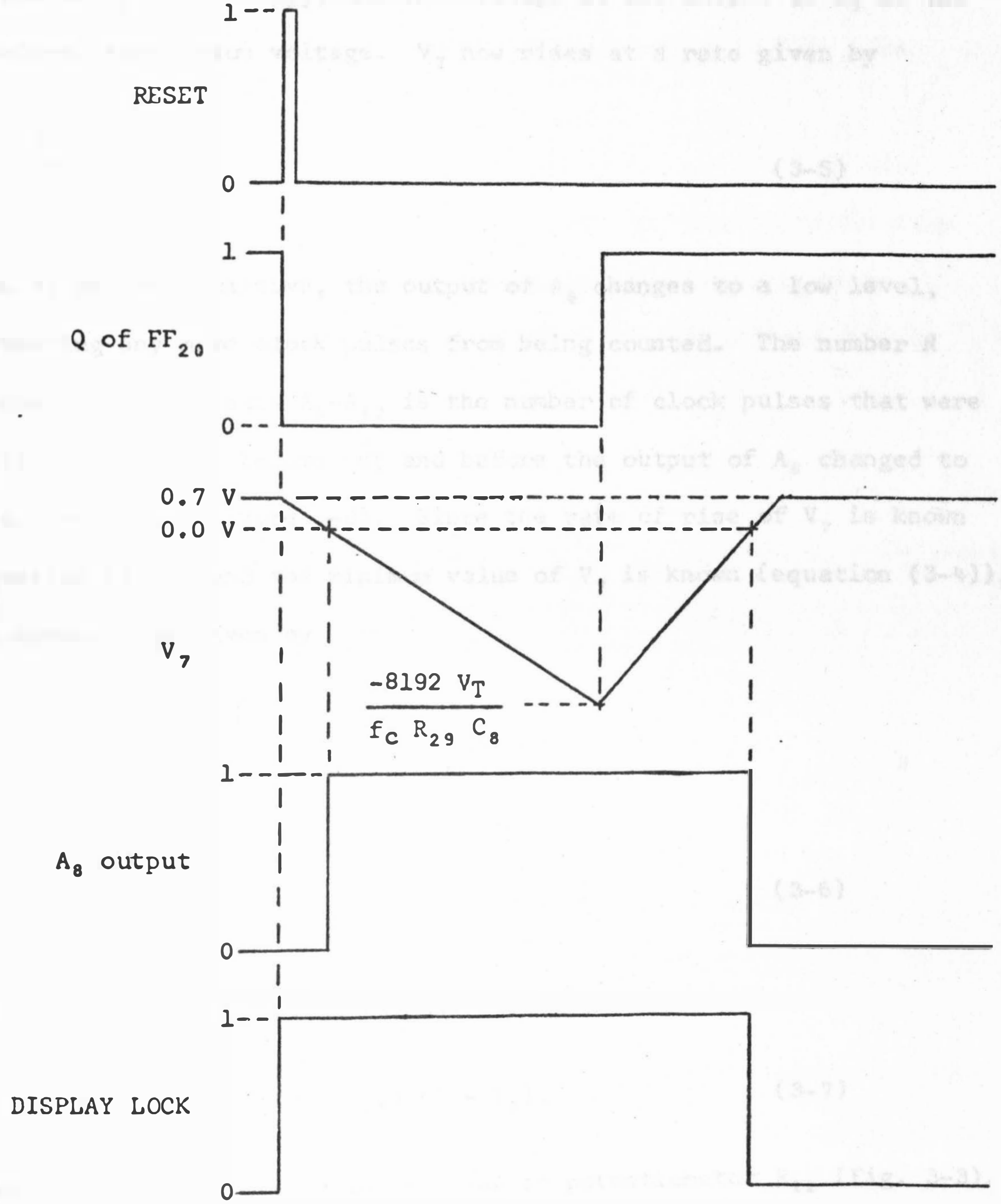

Figure 3-8. Timing Diagram for A/D Converter

where f_c is the clock frequency. The voltage at the open-collector output of I_8 is now V_{ref}, and the voltage at the output of I_9 is the low-level saturation voltage. V_7 now rises at a rate given by

$$\frac{dV_7}{dt} \approx \frac{V_{ref}}{R_{29}\ C_8}. \qquad (3\text{-}5)$$

When V_7 becomes positive, the output of A_8 changes to a low level, preventing any more clock pulses from being counted. The number N represented by outputs A_0-A_{12} is the number of clock pulses that were applied after FF_{20} became set and before the output of A_8 changed to a low level (see Figure 3-8). Since the rate of rise of V_7 is known (equation (3-5)) and the minimum value of V_7 is known (equation (3-4)), the number N is given by

$$N = \left| \frac{V_{7,min}\ f_c}{\frac{dV_7}{dt}} \right|,$$

$$\text{or} \quad N = \frac{2^{13}\ V_T}{V_{ref}}. \qquad (3\text{-}6)$$

The voltage V_T is given in §3.3 as

$$V_T \approx 0.0125\ V_{ref}(R_{16}/R_{15})\ (T - T_0), \qquad (3\text{-}7)$$

where T_0 is determined by the setting of potentiometer R_{12} (Fig. 3-3).

From equations (3-6) and (3-7) the number N represented by the A/D converter outputs is given by

$$N \approx 100\ (R_{16}/R_{15})\ (T - T_0). \qquad (3\text{-}8)$$

Note that (3-8) does not depend upon V_{ref}, f_c, R_{29}, or C_8. It does depend on the temperature T of thermistor R_T, the setting of potentiometer R_{12}, and the gain-determining elements R_{15}-R_{18} (see Figure 3-3, §3.3).

Although values of R_{29} and C_8 do not affect (3-8), they are somewhat critical. Values of V_T are limited to values greater than zero and less than V_{ref} in order for the A/D converter to work properly. The minimum possible value of V_7 is given (from 3-4) by

$$V_7 \text{ min} \simeq -\frac{8192\ V_{ref}}{f_c\ R_{29}\ C_8}$$

To prevent saturation of amplifier A_7, values of R_{29} and C_8 were chosen such that

$$|V_7 \text{ min}| < |V^-|,$$

$$\text{or} \quad R_{29}\ C_8 > \left|\frac{8192\ V_{ref}}{f_c\ V^-}\right|,$$

where V^- is the value of V_7 at which A_7 becomes saturated.

3.5.1 Generation of reset pulse

A monostable multivibrator circuit is shown in Figure 3-9. When power is applied, the $\overline{\text{LOG}}$ $\overline{\text{CLOCK}}$ input is at a high level (§3.4, Figure 3-5). Capacitor C_9 charges through the input of I_{14} until the output of I_{14} becomes a low level, and the output of I_{13} opens. When a LOG CLOCK pulse occurs, the output of I_{12} changes to a high level and

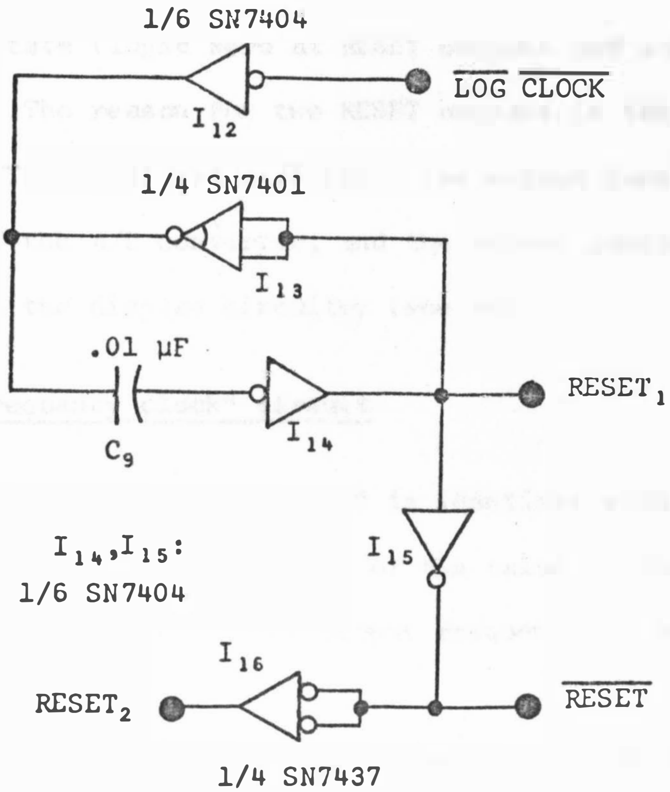

Figure 3-9. Reset Pulse Generator

capacitor C_9 discharges by drawing positive charge from the output of I_{12}. On the falling edge of a LOG CLOCK pulse (a positive transition of the $\overline{\text{LOG CLOCK}}$ input level), the output of I_{12} changes to a low level. Voltage across C_9 cannot change instantaneously, so the RESET outputs change to a high level and the $\overline{\text{RESET}}$ output changes to a logic zero. Capacitor C_9 charges through the input of I_{14} until the outputs return to the stable state (logic zero at RESET outputs and logic one at the $\overline{\text{RESET}}$ output). The reason for two RESET outputs is the limited fan-out capability of TTL integrated circuits. The output labeled RESET_1 resets the counters of the A/D converter, and the output labeled RESET_2 resets the counters in the display circuitry (see §3.7).

3.5.2 "High-frequency clock" circuit

The clock circuit of Figure 3-10 is identical with that of Figure 3-4 (§3.4) with the exception of the value of the capacitor C_{10}. The value of .005 μF causes output frequency to be about 66 kHz.

With capacitor C_9 of Figure 3-9 chosen at .01 μf, the RESET_1 output remains at a high logic level for about 30 μs. The output voltage V_7 of integrator A_7 (Fig. 3-7) is held at about +0.7 V by diode D_2 when the reset pulse occurs. The rate of decrease of V_7 is given above by equation (3-3) as

$$-\frac{dV_7}{dt} \simeq \frac{V_T}{R_{29}\, C_8} \qquad (3\text{-}3)$$

The maximum value of V_T is physically limited by the operational

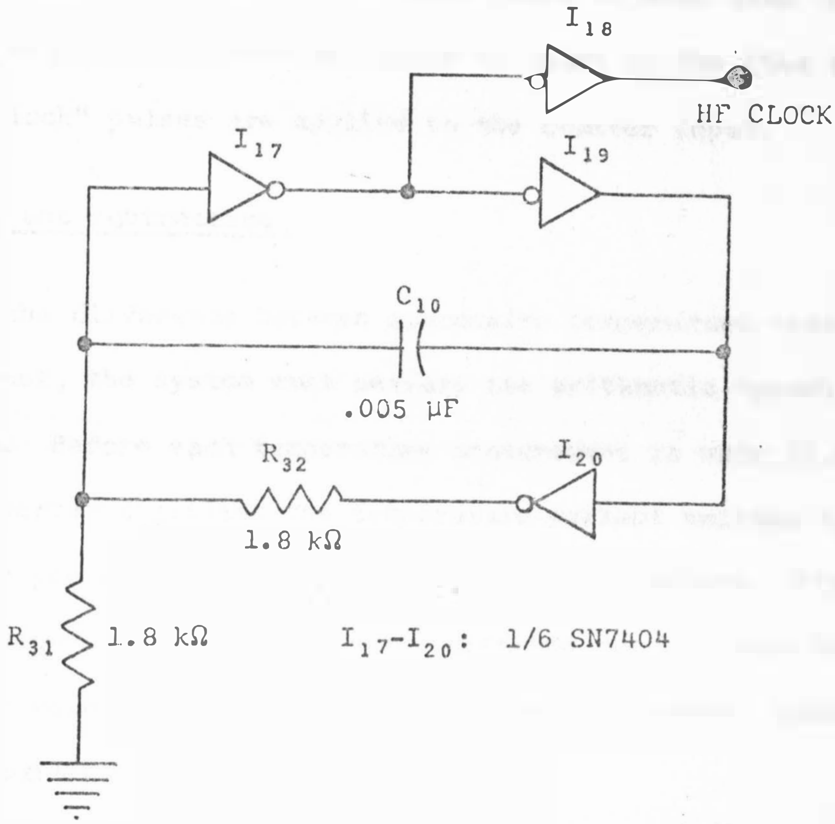

Figure 3-10. High-frequency Clock

amplifier power supply (§3.8) at values less than 9 V. Therefore the minimum time required for V_7 to reach 0.0 V from +0.7 V is given by

$$\frac{(0.7\ \text{V})\ R_{29}\ C_8}{V_T\ \text{max}} = \frac{(0.7\ \text{V})\ (270\ \text{k}\Omega)\ (1\ \mu\text{F})}{9\ \text{V}} = 21\ \text{msec.}$$

Since the 30 μsec duration of the reset pulse is much less than 21 msec, counters C_1-C_3 will be reset and ready to count by the time the "high-frequency clock" pulses are applied to the counter input.

3.6 Memory and subtraction

Since the difference between successive temperature measurements is of interest, the system must perform the arithmetic operation of subtraction. Before each temperature measurement is made (i.e. before the A/D converter digitizes the temperature-variant voltage V_T), the value of the preceding measurement is stored in a memory. The difference between two successive measurements can be found by subtracting the value stored in memory from the more recent value available at the A/D converter output.

However, an identical procedure is to store the binary representation of the negative of the value of earlier measurement. The later measurement available at the A/D converter outputs is then added to the negative quantity stored in memory. The circuit of Figure 3-11 stores the 1's complement of the earlier measurement and uses a sign-magnitude type of adder to find the algebraic sum.

The logic levels at the inputs A_0-A_{12} from the A/D converter

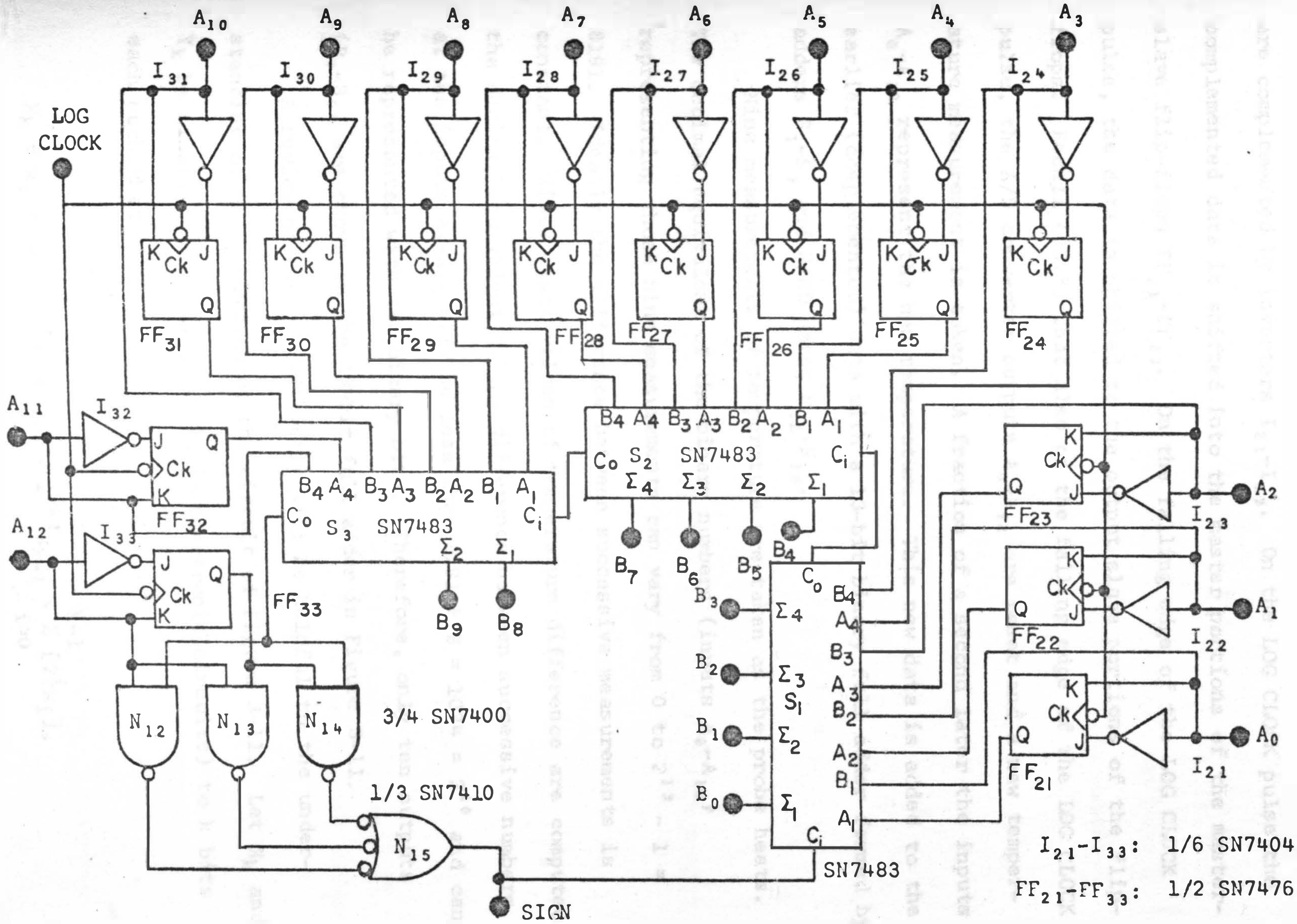

Figure 3-11. Inverter, Memory, and Adder

are complemented by inverters I_{21}-I_{33}. On the LOG CLOCK pulse the complemented data is shifted into the master portions of the master-slave flip-flops FF_{21}-FF_{33}. On the falling edge of the LOG CLOCK pulse, the data is shifted to the output slave portions of the flip-flops. Recall (§3.5) that also on the falling edge of the LOG CLOCK pulse, the A/D converter outputs A_0-A_{12} are reset and a new temperature measurement is taken. A fraction of a second later the inputs A_0-A_{12} represent the new temperature. This new data is added to the earlier (complemented) data with a 13-bit binary full adder formed by adders S_1-S_3 and NAND gates N_{12}-N_{15}.

Nine measurements of temperature are taken as the probe heats. The decimal equivalent of the binary numbers (inputs A_0-A_{12}) representing those nine measurements can vary from 0 to $2^{13} - 1 = 8191$. Ideally the difference between successive measurements is constant. Since eight values of temperature difference are computed, the decimal equivalent of the difference between successive numbers at the inputs A_0-A_{12} will be less than $8191 \div 8 \approx 1024 = 2^{10}$ and can be represented with ten binary bits. Therefore, only ten outputs (B_0-B_9) are shown on the 13-bit full adder in Figure 3-11.

A review of binary arithmetic will be helpful to the understanding of the operation of the circuit of Figure 3-11. Let X_k and Y_k be binary numbers limited (as in electronic circuits) to k bits each such that

$$X_k \equiv x_0 + 2x_1 + 2^2x_2 + \ldots + 2^{k-1}x_{k-1} = \sum_{i=0}^{k-1} [2^i x_i],$$

and $Y_k \equiv y_0 + 2y_1 + 2^2y_2 + \dots + 2^{k-1}y_{k-1} = \sum_{i=0}^{k-1} [2^i y_i]$,

where each x_i and y_i are binary variables (bits) which can have a value of either zero (0) or one (1). A binary number is usually written in the form

$$X_k = x_{k-1}x_{k-2}x_{k-3}\dots x_0.$$

For example, the binary number

$X_4 = 1001$ has bits $x_0 = x_3 = 1$ and bits $x_1 = x_2 = 1$.

X_4 is therefore equivalent to the decimal number 9.

The *one's complement*, or simply *complement*, of a binary number Y_k is defined to be

$$\overline{Y}_k \equiv \sum_{i=0}^{k-1} 2^i \overline{y}_i,$$

where each $\overline{y}_i = 0$ if $y_i = 1$, or $\overline{y}_i = 1$ if $y_i = 0$. Each bit $\overline{y}_i$ is called the *complement* of the bit y_i [Fitchen 1970, p. 263]. It is evident, therefore, that the sum of Y_k and $\overline{Y}_k$ is equal to the number Z_k where each $z_i = 1$, or

$$Y_k + \overline{Y}_k = Z_k = \underbrace{111\dots1}_{k \text{ bits}} \equiv 2^k - 1. \qquad (3\text{-}9)$$

From (3-9) the sum of a k-bit binary number X_k and the complement $\overline{Y}_k$ of another binary number Y_k is given by

$$X_k + \overline{Y}_k \equiv [X_k]_{10} - [Y_k]_{10} - 1 + 2^k, \qquad (3\text{-}10)$$

where $[X_k]_{10}$ and $[Y_k]_{10}$ represent the decimal equivalent of binary numbers X_k and Y_k, respectively.

Suppose that the two numbers X_k and $\overline{Y}_k$ from (3-10) are added with the use of a full adder circuit. Two cases are now to be considered.

First assume that $X_k > Y_k$. Then from (3-10), the sum is at least equal to 2^k. Therefore, the carry-out output of the full-adder is a logical one, representing 2^k. The sum outputs, limited to values less than the binary equivalent of 2^k, represent the other terms of (3-10), or

$$S_k = X_k - Y_k - 1,$$

where $S_k = s_{k-1}s_{k-2}\ldots s_0$ is the binary number at the sum outputs. However, if the carry-out output is tied to the carry-in input, called "end-around carry," then 1 is added to S_k causing the new value of S_k to be given by

$$S_k = X_k - Y_k. \qquad (3\text{-}11)$$

For the second case, assume that $X_k \leq Y_k$. Therefore, the sum $X_k + \overline{Y}_k$ given by (3-10) is at most $2^k - 1$. Therefore, the carry-out output of the full adder is a logical zero. The sum outputs are therefore given by

$$[S_k]_{10} = 2^k - 1 - [Y_k - X_k]_{10}$$

which, from (3-9) is equivalent to

$$[S_k]_{10} = [(\overline{Y_k - X_k})]_{10},$$

or $S_k = \overline{Y_k - X_k}$. (3-12)

In the first case above, the sum outputs represent the magnitude of the difference between X_k and Y_k, and the carry-out output is a logical one. In the second case, the sum outputs represent the one's complement of the magnitude of the difference between X_k and Y_k, and the carry-out output is a logical zero.

The two cases above describe the operation of the 13-bit adder shown in Figure 3-11 where X_{13} is the number available at inputs A_0-A_{12} and $\overline{Y}_{13}$ is the complemented number stored in FF_{21}-FF_{33}. The NAND gates N_{12}-N_{15} produce the carry-out output. In the first case ($X_{13} > Y_{13}$) the difference $X_{13} - Y_{13}$ is positive and the carry-out output is a logical one. But in the second case the difference $X_{13} - Y_{13}$ is negative and the carry-out output is zero. Therefore the carry-out output is called the SIGN indicator in Figure 3-11.

3.7 Decode and display

When the SIGN indicator is a logical one, the common input to exclusive-OR gates E_1-E_{10} in Figure 3-12 is a logic zero. The output of each gate E_1-E_{10} takes the same logic level as its B_i input. However, if the SIGN indicator is zero, inputs B_0-B_9 are complemented by exclusive-OR gates E_1-E_{10}. Therefore, from equations (3-11) and (3-12), the outputs of the exclusive-OR gates

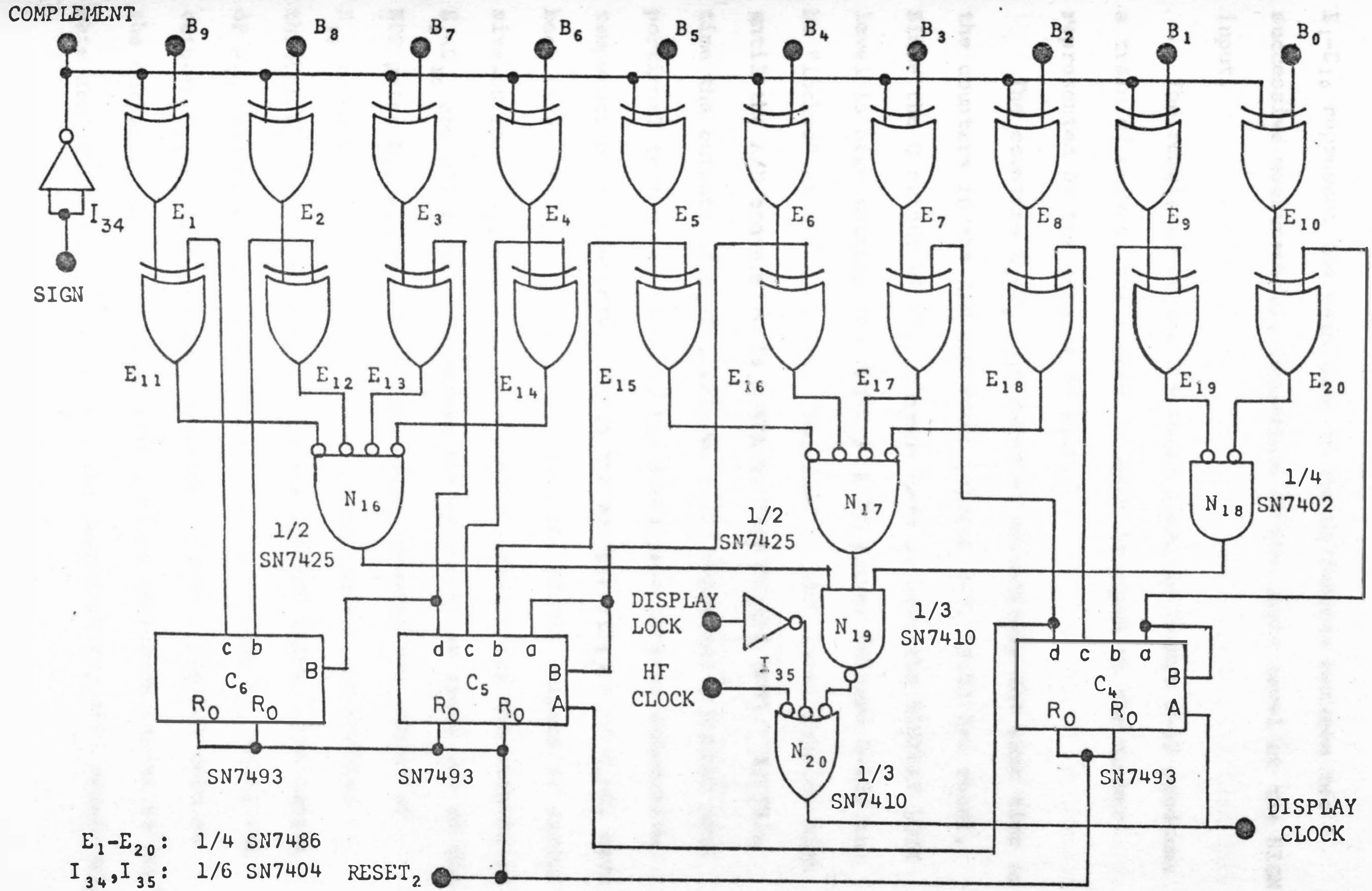

Figure 3-12. 10-bit True/Complement, Binary-to-Serial Converter

E_1-E_{10} represent the magnitude of the difference between two successive measurements, regardless of the logic level at the SIGN input.

The remainder of the circuitry shown in Figure 3-12 produces a train of pulses, the number of which is equal to the number represented by the outputs of E_1-E_{10}.

The counters C_4-C_6 are reset at essentially the same time as the counters in the A/D converter (Figure 3-7, §3.5) are reset. Since the Q output of FF_{20} (Figure 3-7) is low, the DISPLAY LOCK level is high causing the DISPLAY CLOCK output (Figure 3-12) to be "locked" at a high level. The DISPLAY LOCK level remains high until the A/D conversion is complete (see Figure 3-8). At this time the outputs of E_1-E_{10} (Figure 3-12) represent a number proportional to the magnitude of the difference between successive temperature measurements. Since the counter outputs of C_4-C_6 have been reset to logical zeros, at least one of the outputs of exclusive-OR gates E_{11}-E_{20} will be a logical one--unless the outputs of E_1-E_{10} are all zeros. Therefore the output of at least one of the NOR gates N_{16}-N_{18} will be a logic zero, causing the output of N_{19} to be a logical one. Therefore clock pulses are enabled through N_{20} to the counter C_4 and the DISPLAY CLOCK. The output of N_{19} remains high, allowing clock pulses through N_{20} until the outputs E_{11}-E_{20} are all logical zeros. Since a logical zero at the output of an exclusive-OR gate implies that both inputs to the gate are at the same logic level, the output of N_{20} will remain high when the number represented by the outputs of C_4-C_6 equals the number represented by the outputs of E_1-E_{10}.

The counters C_8-C_{10} of Figure 3-13 are reset at the same time as those of Figure 3-12. The UNITS, TENS, and HUNDREDS displays all show the numeral "0". After each A/D conversion, clock pulses are present at the DISPLAY CLOCK input. As the outputs of counters C_4-C_6 of Figure 3-12 count up to the binary equivalent of the number represented by the outputs of E_1-E_{10}, the counters C_8-C_{10} count up to the binary-coded-decimal (BCD) equivalent of the same number. The numbers on the display therefore count from zero to a number proportional to the magnitude of the difference between successive temperature measurements. By controlling the gain elements of the temperature sensing amplifier (§3.3) and the A/D converter (§3.5), the number displayed can be made equal to thermal resistivity (in cm-sec-°C/10^{-1}cal) of the material surrounding the cylindrical probe.

As shown in Figure 3-13, if the COMPLEMENT input is at a high level the decimal points in the 3-digit thermal resistivity display are lit as an indication of the negative result.

As power is applied, counter C_7 outputs are set to BCD nine, so the numeral "9" is displayed on the COUNT display. After the first LOG CLOCK (Figure 3-5, §3.4) pulse the COUNT display changes to "0". The first measurement of temperature difference on thermal resistivity occurs after the second LOG CLOCK pulse, when the COUNT display changes to "1". The COUNT display therefore displays the

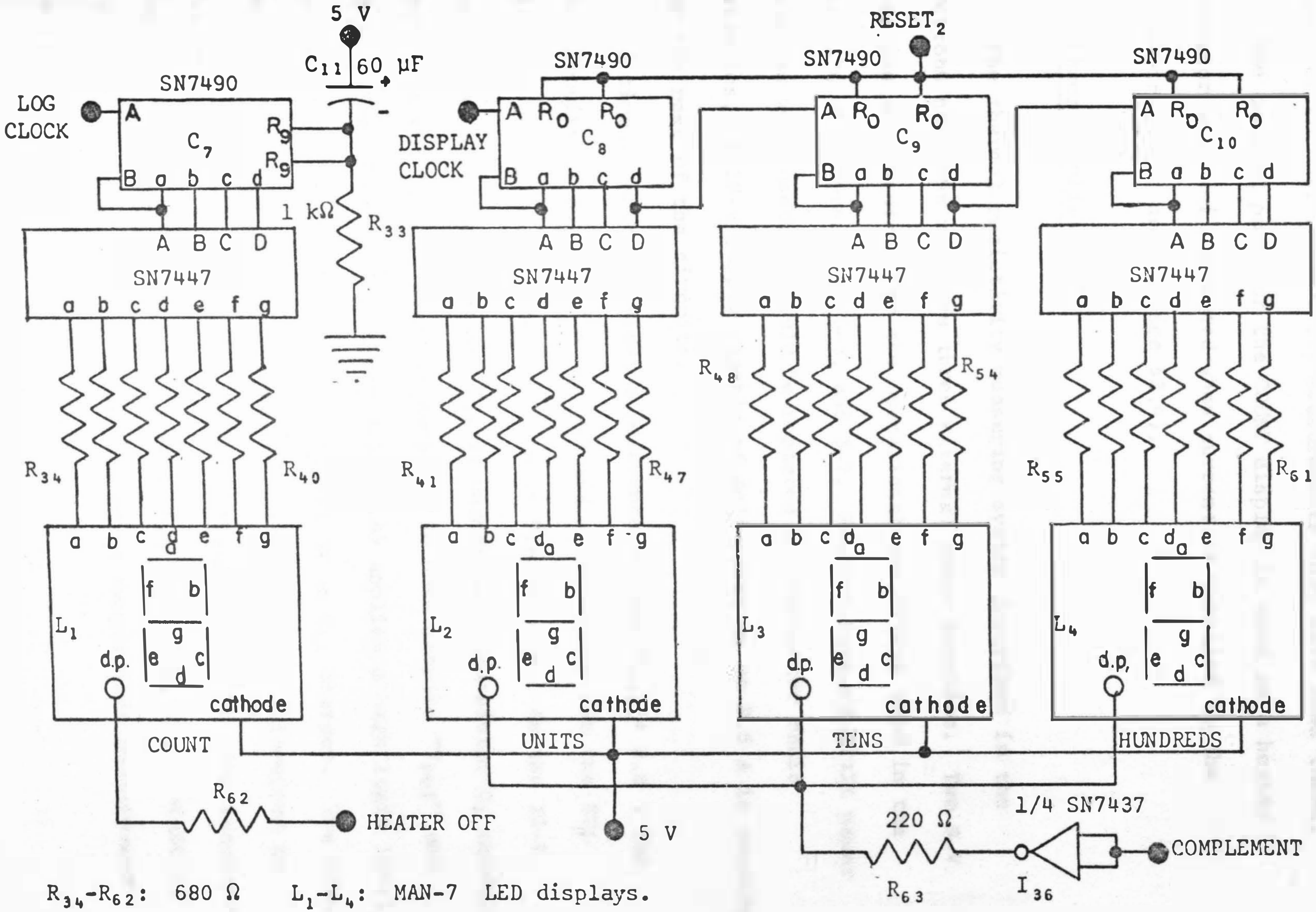

Figure 3-13. Display Counters, Decoder-Drivers, and LED Displays

number of thermal resistivity measurements that have been taken.

The decimal point of the COUNT display is used as a heater indicator and is illuminated when current is supplied to the cylindrical heat source (see §3.4).

3.8 Power supplies

The thermal resistivity measuring system described in the previous sections requires three external power supplies. Two 9-V supplies are required for the operational amplifiers used in the circuits of Figures 3-2, 3-3, and 3-7. Hewlett-Packard #741A power supplies were used but could be replaced by transistor radio batteries. A 12-V supply capable of delivering up to 2.5 A is required for the rest of the circuits.

A circuit which provides a reference voltage V_{ref} = 6.8 V for the circuits of Figures 3-3 and 3-7 and a 5-V supply for the TTL circuits and display diodes is shown in Figure 3-14. As the 12-V supply is turned on capacitor C_{12} is uncharged. Transistor Q_2 conducts and Q_3 is cut off, so power is supplied to both outputs "V_{ref}" and "5 V." The circuitry of Figure 3-5 (§3.4) applies a high logic level to the POWER ON input, keeping Q_2 biased ON as C_{12} charges. The time required for V_{ref} to reach zener voltage of 6.8 V is determined by R_{66} and C_{12}. From the values shown in Figure 3-14, the time required for V_{ref} to reach a value within 2% of 6.8 V is $4 R_{66} C_{12} = 4(100\ \Omega)$ $(0.47\ \mu F) \simeq 0.2$ msec, well before the first temperature measurement is taken.

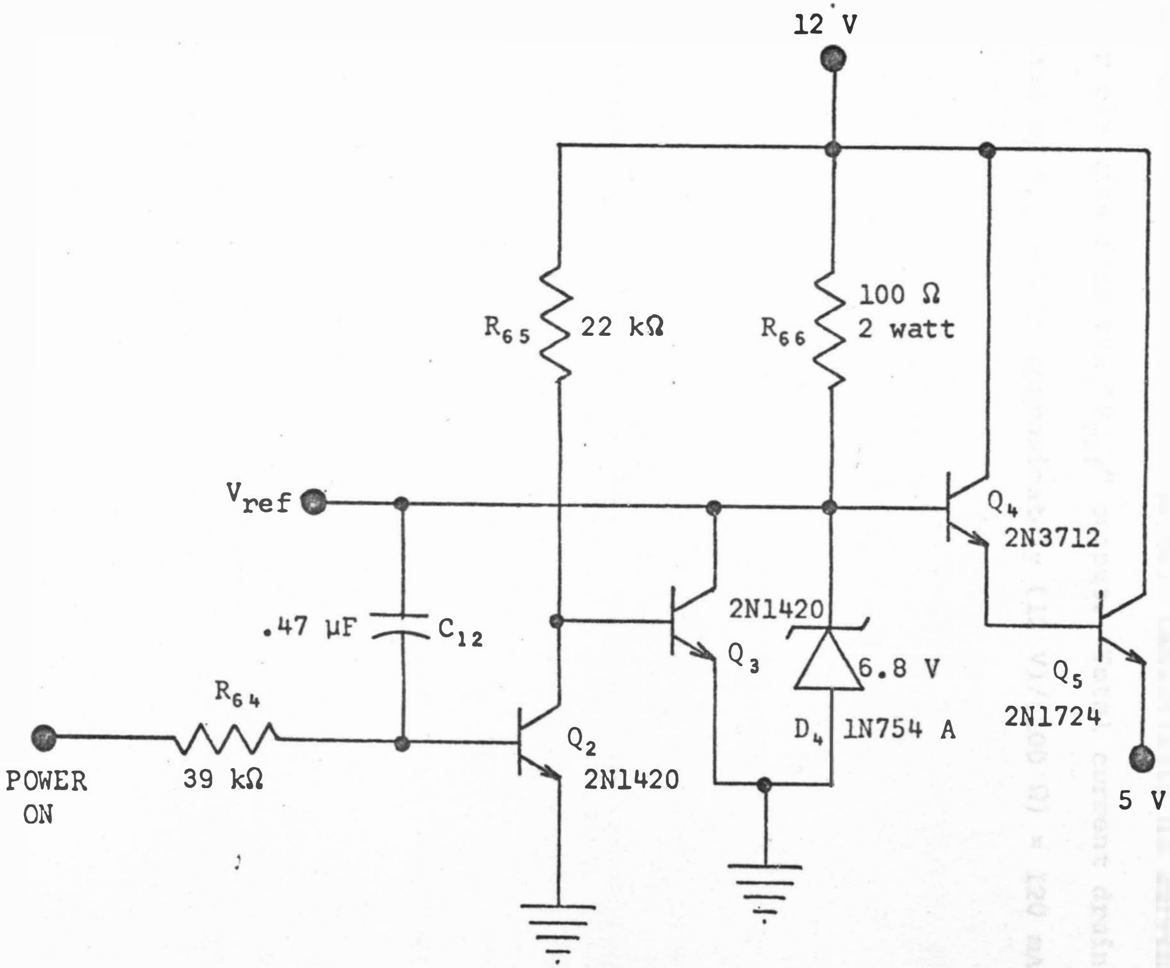

Figure 3-14. TTL and Reference Voltage Supplies

If a low logic level is applied to the POWER ON input, Q_2 is cut off, causing V_{ref} to drop to the collector-emitter saturation voltage of Q_3, causing Q_4 and Q_5 to be cut off. No current is supplied through the "5 V" output in this state. Essentially no current flows to other circuits from the "V_{ref}" output. Total current drain is determined by R_{66} and is approximately (12 V)/(100 Ω) = 120 mA.

CHAPTER FOUR

RESULTS, CONCLUSIONS AND RECOMMENDATIONS

4.1 Results

In this chapter the results of measurements of the thermal resistivity of a fine white sand are given. Results are given for three different moisture contents of the sand. Conclusions and recommendations are given in §4.2 and §4.3, respectively.

4.1.1 System Operation

A summary of the operation of the proposed system to measure thermal resistivity is found in §3.1. The probe heat source of the system supplies a constant rate of heat q per unit time per unit length to the surrounding medium (sand or soil) for a limited time τ. If τ is large, the results of Chapter 2 indicate that measurements of changes in temperature $T_i - T_{i-1}$ over a constant logarithmic time interval $(\ln[t_i] - \ln[t_{i-1}] = \text{constant})$ will yield constant results for large $t_i < \tau$, or (from (2-32))

$$T_i - T_{i-1} = (q/4\pi K)\,[\ln(t_i) - \ln(t_{i-1})]. \qquad (4\text{-}1)$$

Since the system measures temperature at times $t_1, t_2, \ldots, t_i, \ldots$ such that t_i/t_{i-1} is constant (=2), the difference $\ln(t_i) - \ln(t_{i-1}) = \ln(t_i/t_{i-1})$ is constant, and the difference $T_i - T_{i-1}$ between successive temperature measurements is therefore expected to be constant for large $t_i < \tau$.

The case in which the heat source ceases to produce heat after a finite time τ has been investigated by de Vries and Peck [1958a]. The solution for temperature for times much larger than τ is given by

$$T = (q/4\pi K) \ln[t/(t - \tau)]. \tag{4-2}$$

Therefore, the difference $T_k - T_{k-1}$ between successive temperature measurements at times t_{k-1} and t_k ($t_k > t_{k-1}$) is given by

$$T_k - T_{k-1} = \frac{q}{4\pi K} \ln\left[\frac{t_{k-1} - \tau}{t_k - \tau} \frac{t_k}{t_{k-1}}\right], \quad t_{k-1} >> \tau. \tag{4-3}$$

Equation (4-3) can be rewritten in terms of the number of clock pulses n_k that have been applied to the input of the Log Clock circuit of §3.4 at the time of measurement of temperature T_k:

$$T_k - T_{k-1} = \frac{-q}{4\pi K} \ln\left[\frac{n_k - 256}{n_{k-1} - 256} \frac{n_{k-1}}{n_k}\right]. \tag{4-4}$$

With the system described in Chapter 3, the heat source is turned off after time τ = 256 clock pulses. Measurements of temperature T_k are then taken at times t_k (on clock pulses n_k) such that

$$\frac{t_k - \tau}{t_{k-1} - \tau} = \frac{n_k - 256}{n_{k-1} - 256} = 2. \tag{4-5}$$

The difference $T_k - T_{k-1}$ between successive temperature measurements, from (4-4), is not expected to remain constant, but rather to decrease in magnitude by a factor depending upon the ratio n_{k-1}/n_k. The data provided by the system during the "cooling portion" is still of use in finding thermal resistivity, however. If each measurement of temperature difference is multiplied times the appropriate "correction factor" found in the fourth column of Table 4-1, the results should (ideally) all be the same, because use of the correction factor eliminates the dependence of the measurement upon the ratio n_{k-1}/n_k. Thus, the only factor left (on the right side of equation (4-4)) which can cause changes in measurements of temperature difference is 1/K, the thermal resistivity.

4.1.2 Measurements in fine silica sand

For measuring thermal resistivity, values of $R_{15} = R_{18} = 820\ \Omega$ and $R_{16} = R_{17} = 39\ k\Omega$ (Figure 3-3) were chosen. From equation (3-8) of §3.5 it can be shown that a change of .01° C produces a change of about 50 in the number at the output of the A/D converter.

The number displayed by the system represents a change in temperature from one measurement to the next. However, everything on the right-hand side of equation (4-1) (which holds for large values of time t), except thermal resistivity 1/K, is held constant by the measuring system for $t < \tau$. Therefore, the system displays an indication of thermal resistivity in *thermal resistivity units* (TRU).[1]

[1] It was mentioned in Chapter 3 that the system could be adjusted to display thermal resistivity in cm-sec-°C/10^{-1}cal. By adjusting the gain elements R_{15}-R_{18}, this can be accomplished. However, a suitable solid, perhaps paraffin, having a known thermal resistivity, must be used to calibrate the system. The *thermal resistivity unit* (TRU) is a unit of thermal resistivity which must simply be multiplied by a constant to obtain the CGS equivalent.

TABLE 4-1

Correction Factors for Measurements

during the Cooling Portion

k	n_k	n_{k-1}/n_k	$\ln(2)/\ln(2\, n_{k-1}/n_k)$
9	257	---	---
10	258	.9961	1.0056
11	260	.9923	1.0113
12	264	.9848	1.0225
13	272	.9706	1.0450
14	288	.9444	1.0899
15	320	.9000	1.1792
16	384	.8333	1.3569

For the cooling portion, equation (4-3) applies. The correction factors of Table 4-1 must be applied to the data supplied by the system in order for the data to depend upon thermal resistivity alone.

Results of the measurements of the thermal resistivity of a fine silica sand are shown in Figures 4-1 and 4-2. The data is tabulated in Appendix III. The values of moisture content given in the figures were found by the formula

$$(W_W/W_S)\ 100\%, \tag{4-6}$$

where W_S is the weight of the dry sand and W_W is the weight of the water contained in the sand.

4.2 Conclusions

The later measurements in each of Figures 4-1 and 4-2 are assumed to be more valid than the earlier ones, since equations (4-1) and (4-3) are valid only for the later measurements.

A significant increase in the last measurement over the others during the heating period is possibly due to convection. In moist materials, water tends to move away from the source of heat [de Vries and Peck 1958b]. The dryer sand near the probe has a higher thermal resistivity, and consequently the higher reading is obtained. It is supposed that convection effects are negligible over most of the time of measurement, but it is possible that convection was the cause of the increase in the measurements of the *moist* sand. However, it does not seem plausible that this argument can explain the increase in the final measurement of the thermal resistivity of the *dry* sand.

MEASUREMENTS OF THERMAL RESISTIVITY

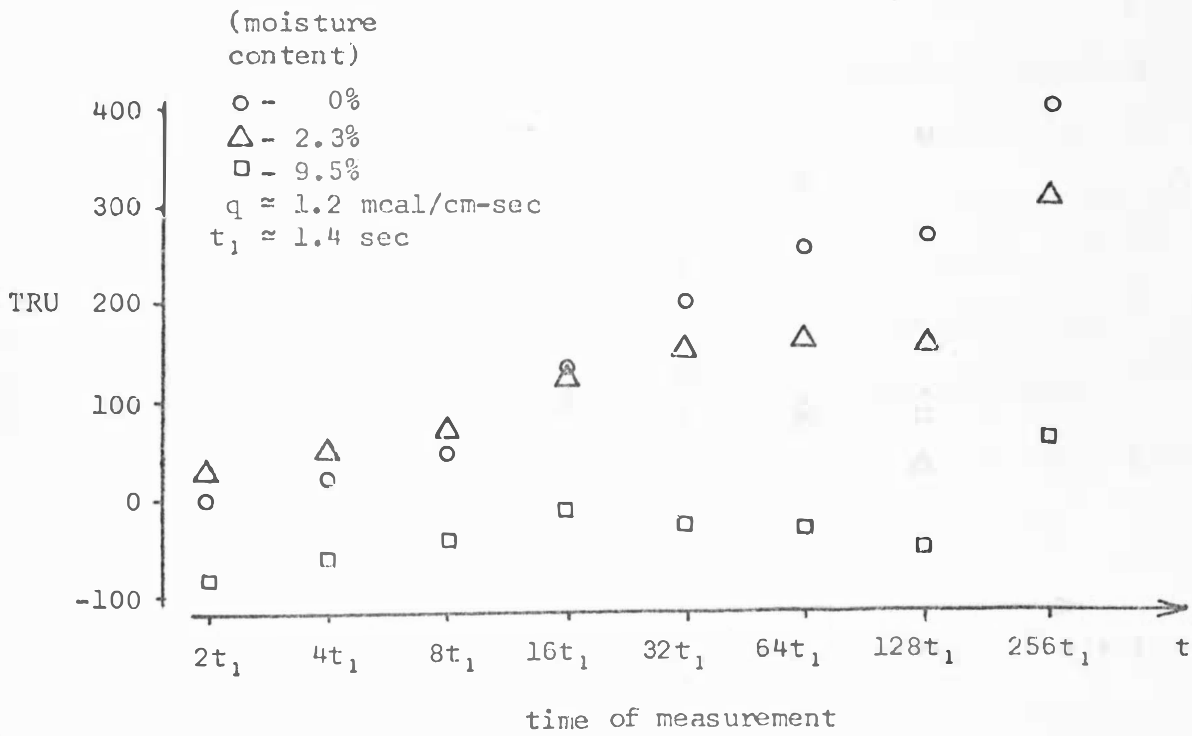

Figure 4-1. Results during Heating Period

MEASUREMENTS OF THERMAL RESISTIVITY

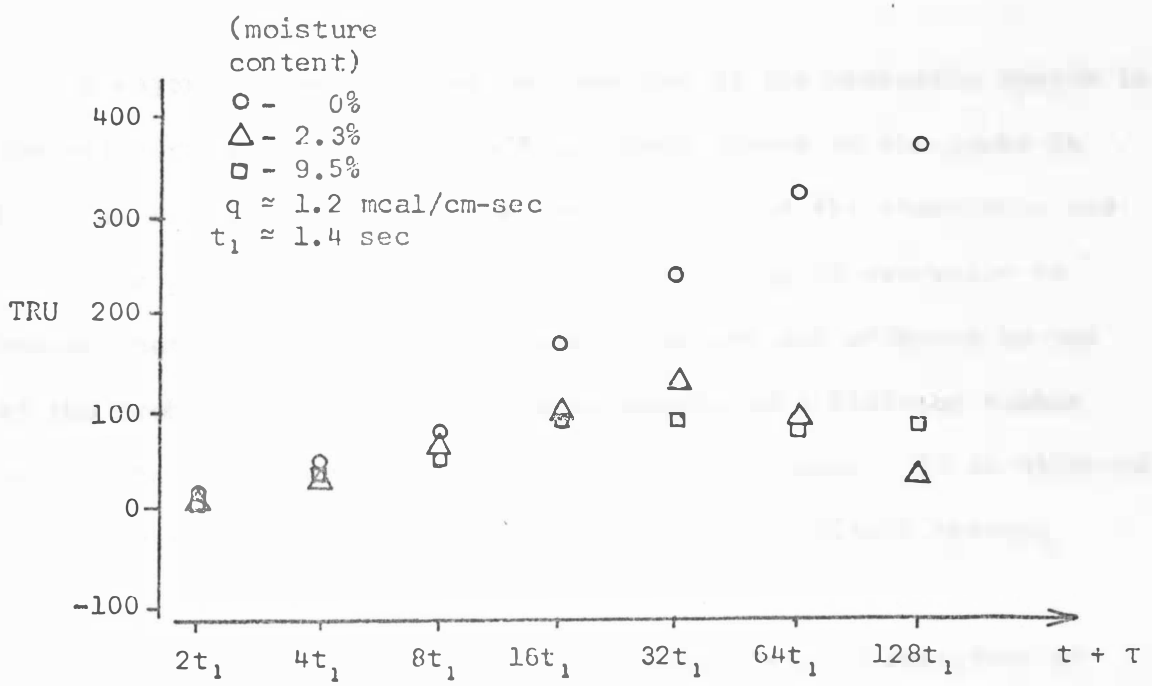

Figure 4-2. Results during Cooling Period

While it is difficult to determine the actual value of thermal resistivity from the above data, it is evident that the data provides an indication of the amount of moisture contained within the sand. With refinement, the system may find application in determination of soil moisture in the field.

4.3 Recommendations

A major problem with the construction of the measuring system is the cylindrical probe. (A cross-sectional sketch of the probe is shown in Figure 3-1.) Adequate waterproofing of the thermistor and its leads proved to be a problem. Waterproofing is necessary to ensure that the thermistor characteristics are not affected by use of the probe in moist soils. A thin coating of a silicone rubber around the bead thermistor seems to work fairly well. It is believed that better probe construction will yield more reliable thermal resistivity measurements.

In general, the gain of the temperature-sensing amplifier of Figure 3-3 is high. Invariably, voltage V_T is not between 0 V and V_{ref} at the time a sequence of measurements is begun. For correct operation of the A/D converter (§3.5), potentiometer R_{12} must be adjusted to bring V_T within the limits mentioned. A provision should be made to allow adjustment of the initial value of V_T before the probe heater is turned on.

As mentioned above, the results indicate that the thermal resistivity of porous materials such as sand and soils is determined to a large extent by the moisture content. It is hoped that further research can be done to determine the extent to which a measuring system such as the one described can be used in the determination of moisture content. Since the system is portable, it would seem likely that it might be useful in field measurements.

The system, as designed, requires an operator to record the data as it is displayed. Since a TTL logic pulse occurs on each measurement of thermal resistivity, it seems possible that an automatic recording system could be devised that records the data on each pulse. The resistivity data is available in a variety of forms including serial, parallel-BCD, or parallel-7-segment.

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NON-REFERENCED MATERIAL

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APPENDIX I

Temperature in an Infinite Solid Bounded Internally by an Infinite Line Source of Heat

Taking the line source as the z-axis, the applicable equations are (from §2.2)

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}, \quad r > 0,\ t > 0; \tag{I-1a}$$

$$T = 0, \quad t = 0; \tag{I-1b}$$

$$\lim_{r\to 0+} r\frac{\partial T}{\partial r} = \frac{-q}{2\pi K}, \quad t > 0; \tag{I-1c}$$

and $$\lim_{r\to\infty} (T) = 0, \quad t > 0. \tag{I-1d}$$

To derive a solution to the above boundary value problem, the Laplace transformation of each of the above equations is used. The Laplace transform of the temperature T, denoted by $\overline{T}$, is defined by

$$\overline{T} = \int_0^\infty \exp(-st)\, T(r,t)\, dt.$$

Upon application of the Laplace transformation to equations (I-1), one obtains the following set:

$$\frac{d^2\overline{T}}{dr^2} + \frac{1}{r}\frac{d\overline{T}}{dr} = \frac{s}{\alpha}\overline{T}, \quad r > 0; \tag{I-2a}$$

$$\lim_{r\to+\infty} \overline{T} = 0; \tag{I-2b}$$

and $\lim_{r \to 0+} r \frac{d\overline{T}}{dr} = \frac{-1}{s} \frac{q}{2\pi K}$. (I-2c)

Equation (I-2a) is recognized as the modified Bessel equation of order zero with parameter $\sqrt{s/\alpha}$. A complete solution is given by [Spiegel 1968, p. 139; Wylie 1966, p. 359]

$$\overline{T} = c_1 I_0(r\sqrt{s/\alpha}) + c_2 K_0(r\sqrt{s/\alpha}), \qquad \text{(I-3)}$$

where I_0 is known as the modified Bessel function of the first kind of order zero, and K_0 is the modified Bessel function of the second kind of order zero. Constants c_1 and c_2 are to be determined from boundary conditions (I-2b) and (I-2c).

Equation (I-2b) requires that the solution (I-3) vanish (approach zero) as r becomes infinite. Since $I_0(x)$ becomes infinite as x approaches infinity, constant c_1 in equation (I-3) must be zero. The solution, still subject to condition (I-2c), now becomes

$$\overline{T} = c_2 K_0(r\sqrt{s/\alpha}). \qquad \text{(I-4)}$$

Application of condition (I-2c) to the solution (I-4) requires that

$$\lim_{r \to 0+} \left\{ r\, c_2 \frac{d}{dr}[K_0(r\sqrt{s/\alpha})] \right\} = -\frac{1}{s} \frac{q}{2\pi K}$$

or $c_2 \lim_{r \to 0+} [r\sqrt{s/\alpha}\, K_1(r\sqrt{s/\alpha})] = q/2\pi K s$. (I-5)

Consider the argument $x \cdot K_1(x)$ of the limit operation in equation (I-5), where $x = r\sqrt{s/\alpha}$. Expanding the modified Bessel function of the second kind of order one (K_1) results in [Spiegel 1968, p. 139]

$$\lim_{x \to 0+} [x\, K_1(x)] = 1 + \lim_{x \to 0+} \left\{ \sum_{k=1}^{\infty} \frac{x^{2k}\, [2 \ln(Ex/2) - \Phi(k) - \Phi(k-1)]}{2^{2k}\, (k-1)!\, k!} \right\}$$

where $\Phi(k) = \sum_{i=1}^{k} (1/i)$, $\Phi(0) = 0$, and $\ln(E) = .5772...$ is Euler's constant. Since terms of order x^{2k} vanish as $x \to 0$, this expression reduces to

$$\lim_{x \to 0+} [x\, K_1(x)] = 1 + \lim_{x \to 0+} \left\{ \sum_{k=1}^{\infty} \frac{2 \ln(x)}{(2/x)^{2k}\, (k-1)!\, k!} \right\}$$

which, upon reversing the limit and summation operations and application of l'Hospital's Rule, becomes

$$\lim_{x \to 0+} [x\, K_1(x)] = 1 - \sum_{k=1}^{\infty} \lim_{x \to 0+} [(x/2)^{2k}/(k!)^2] = 1.$$

Therefore, the constant c_2 in equation (I-5) is given by

$$c_2 = \frac{1}{s} \frac{q}{2\pi K}, \qquad \text{(I-6)}$$

and the solution to equations (I-2) is, from equations (I-4) and (I-6),

$$\overline{T} = \frac{1}{s} \frac{q}{2\pi K} K_0(r\sqrt{s/\alpha}). \qquad \text{(I-7)}$$

The solution to equations (I-1) is found by taking the inverse Laplace transform of equation (I-7). First, however, consider the inverse transform of $K_0(r\sqrt{s/\alpha})$, given by [Carslaw and Jaeger 1959,

p. 495; Bateman 1954, p. 283]

$$(1/2t)\,\exp(-r^2/4\alpha t).$$

Division by the constant $2\pi K/q$ and integration with respect to time t yields an expression for temperature T given by

$$T = (q/4\pi K)\int_0^t (1/t)\exp(-r^2/4\alpha t)\,dt$$

$$= (q/4\pi K)\int_{(r^2/4\alpha t)}^{\infty} [\exp(-u)/u]\,du$$

$$= -\,(q/4\pi K)\,\mathrm{Ei}(-r^2/4\alpha t),\quad t > 0. \qquad \text{(I-8)}$$

The solution (I-8) can be expressed in series form [Spiegel 1968, p. 183] as

$$T = (q/4\pi K)\,[\ln(t) - \ln(r^2 E/4\alpha) - \sum_{k=1}^{\infty} \{(-r^2/4\alpha t)^k / k\cdot k!\}].$$

APPENDIX II

Temperature in an Infinite Solid Bounded Internally by a Right Circular Cylinder with Constant Heat Flux at its Surface

The same initial and boundary conditions as in Appendix I are again considered here, except that condition (I-1c) is now

$$\lim_{r\to r_1+} r\frac{\partial T}{\partial r} = \frac{-q}{2\pi K}, \quad t > 0,$$

where r_1 is the radius of the right circular cylinder whose axis is the z-axis. The applicable equations for deriving a solution are

$$\frac{d^2\overline{T}}{dr} + \frac{1}{r}\frac{d\overline{T}}{dr} = \frac{s}{\alpha}\overline{T}, \quad r > r_1; \tag{II-1a}$$

$$\lim_{r\to+\infty} \overline{T} = 0; \tag{II-1b}$$

and

$$\lim_{r\to r_1+} r\frac{d\overline{T}}{dr} = \frac{-1}{s}\frac{q}{2\pi K}. \tag{II-1c}$$

As in Appendix I the solution to (II-1a) subject to condition (II-1b) is of the form

$$\overline{T} = c\cdot K_0(r\sqrt{s/\alpha}), \tag{II-2}$$

where c is yet to be determined. Subjecting equation (II-2) to condition (II-1c) yields

$$c \lim_{r\to r_1+} [r\sqrt{s/\alpha}\, K_1(r\sqrt{s/\alpha})] = q/2\pi Ks,$$

or $\quad c = q \,/\, [2\pi r_1 \; K \; s\sqrt{s/\alpha} \; K_1(r_1\sqrt{s/\alpha})]$.

Therefore the solution becomes

$$\bar{T} = \frac{q\; K_0(r\sqrt{s/\alpha})}{2\pi r_1 \; K \; s\sqrt{s/\alpha}\; K_1(r_1\sqrt{s/\alpha})}, \; r > r_1. \qquad \text{(II-3)}$$

The inverse Laplace transform of (II-3) is given by [Carslaw and Jaeger 1959, p. 338]

$$T = -(q/\pi^2 K r_1) \int_0^\infty \frac{[1 - \exp(-\alpha u^2 t)][J_0(ur)Y_1(ur_1) - Y_0(ur)J_1(ur_1)]}{u^2[J_1^2(ur_1) + Y_1^2(ur_1)]} du$$

(II-4)

where J_n is the Bessel function of the first kind of order n, and Y_n is the Bessel function of the second of order n.

A series solution equivalent to (II-4) may be derived [Carslaw and Jaeger 1959, pp. 340-341] by considering the series expansions of the modified Bessel functions of equation (II-3). In what follows a series expansion of the ratio $[K_0(x)]/[y\; K_1(y)]$ will be developed, and the solution will be found by inverse Laplace transformation of each term.

The series expansions of the modified Bessel functions of the second kind of orders zero and one, respectively, are given by [Spiegel 1968, p. 139]

$$K_0(x) = -\ln(Ex/2) - \sum_{k=1}^{\infty} [(x/2)^{2k}(\ln[Ex/2] - \Phi[k])/(k!)^2], \quad \text{(II-5)}$$

and $$K_1(y) = \frac{1}{y} + \sum_{k=1}^{\infty} \frac{y^{2k-1}\,[2\ln(Ey/2) - \Phi(k) - \Phi(k-1)]}{2^{2k}\,(k-1)!\,k!}$$

where $\Phi(k)$ and $\ln(E)$ are as defined in Appendix I.

The ratio $1/[y\,K_1(y)]$ is therefore given by

$$\frac{1}{y\,K_1(y)} = \frac{1}{1+S} \qquad \text{(II-6)}$$

where $S = \sum_{k=1}^{\infty} [(y/2)^{2k}\,(2\ln[Ey/2] - \Phi[k] - \Phi[k-1])/(k-1)!\cdot k!]$.

Equation (II-6), under certain conditions [Spiegel 1968, p. 110] to be discussed later, can be manipulated by the use of a special case of the binomial expansion to yield

$$(1+S)^{-1} = 1 - S + S^2 - S^3 + \ldots \qquad \text{(II-7)}$$

Multiplication of (II-7) times $-K_0(x)$ (see (II-5)), yields a series. If terms of the order of x^4, x^2y^2, and y^4 and higher are truncated, the series reduces to

$$-K_0(x) / [y\,K_1(y)] \simeq \ln(Ex/2) + (x/2)^2\,[\ln(Ex/2) - 1] - (y/2)^2\,\ln(Ex/2)\,[2\ln(Ey/2) - 1]. \qquad \text{(II-8)}$$

Substitution of equation (II-8) into (II-3) results in an expression of the form

$$\overline{T} \approx (-q/4\pi Ks)\ \{\ln(E^2r^2s/4\alpha) + (r^2s/4\alpha)[\ln(E^2r^2s/4\alpha) - 2]$$
$$- (r_1^2s/4\alpha)\ \ln(E^2r^2s/4\alpha)\ [\ln(E^2r_1^2s/4\alpha) - 1]\},$$

which may be rewritten as

$$\overline{T} \approx (q/4\pi K)\ \{(r^2/2\alpha) - \ln(E^2r^2s/4\alpha)\ [1/s + (r^2 + r_1^2)/4\alpha]$$
$$+ (r_1^2/4\alpha)\ [\ln(E^2r^2s/4\alpha)]^2$$
$$+ (r_1^2/4\alpha)\ \ln(E^2r^2s/4\alpha)\ [2\ln(r_1/r)]\}. \qquad \text{(II-9)}$$

By taking the inverse Laplace transform of each term, one obtains [Spiegel 1968, p. 169; Carslaw and Jaeger 1959, pp. 340-341, 496]

$$T \approx (q/4\pi K)\ \{\ln(4\alpha t/Er^2) + (r^2 + r_1^2)/4\alpha t + (r_1^2/2\alpha t)\ \ln(4\alpha t/Er^2)$$
$$- (r_1^2/2\alpha t)\ \ln(r_1/r), \quad r > r_1,\ t > 0. \qquad \text{(II-10)}$$

The approximation necessary for equation (II-7) and for the truncation of higher orders of the series (II-8) is that the quantity sr^2/α is small. This approximation holds for large values of time t (much greater than r^2/α).

For points near the cylinder ($r \approx r_1$) and for large values of time t, solution (II-10) reduces to

$$T \approx (q/4\pi K)\ \{\ln(4\alpha t/Er^2) + 0(\ln[t]/t)\} \qquad \text{(II-11)}$$

APPENDIX III

Measurements of Thermal Resistivity

The data from thermal resistivity measurements of a fine silica sand is contained in Tables III-1, -2, and -3. Measurements were taken of a dry sand, a damp sand (2.3% moisture content by weight), and a wet sand (9.5% moisture content). Three series of measurements were taken at each moisture level.

The sand was contained in plexiglass containers approximately 12 inches (30 cm) long, 11 inches (28 cm) wide, and 3 inches (8 cm) high. The probe was buried in the sand about 1 1/2 inches (4 cm) below the surface.

Considerable scatter of data is observed. Table III-4 shows two series of measurements by the system with the thermistor replaced by a resistor (of constant value). The results are, of course, expected to be approximately zero, since no change in resistance is expected. The data of Table III-4 therefore gives an indication as to the amount of scatter of measurements of thermal conductivity. The average value of scatter is -1.2. The standard deviation is about 16.

TABLE III-1

Thermal Resistivity Measurement of Dry Fine Silica Sand

	MEASUREMENT NUMBER (§3.7)	TRIAL 1 (TRU)	TRIAL 2 (TRU)	TRIAL 3 (TRU)	AVERAGE (TRU)
HEATING PERIOD	1	-4	25	-18	1
	2	23	18	23	21
	3	46	50	53	50
	4	119	153	135	136
	5	174	193	238	202
	6	218	265	297	260
	7	240	266	323	276
	8	352	380	483	405
COOLING PERIOD	10	23	23	4	17
	11	45	58	53	52
	12	88	85	69	81
	13	162	167	165	165
	14	237	213	211	220
	15	292	278	260	277
	16	298	299	238	278
CORRECTED COOLING PERIOD DATA	10	23	23	4	17
	11	46	59	54	53
	12	90	87	71	83
	13	164	175	172	172
	14	258	232	230	240
	15	344	328	307	326
	16	404	406	323	378

TABLE III-2

Thermal Resistivity Measurements of Damp Silica Sand

of Moisture Content 2.3%

	MEASUREMENT NUMBER (§3.7)	TRIAL 1 (TRU)	TRIAL 2 (TRU)	TRIAL 3 (TRU)	AVERAGE (TRU)
HEATING PERIOD	1	26	30	36	31
	2	46	51	49	49
	3	73	76	76	75
	4	121	127	126	125
	5	152	161	153	155
	6	165	170	152	162
	7	163	153	157	158
	8	315	329	298	314
COOLING PERIOD	10	1	10	15	9
	11	23	39	30	31
	12	61	61	56	59
	13	94	100	103	99
	14	124	118	109	117
	15	75	74	90	80
	16	20	47	12	26
CORRECTED COOLING PERIOD DATA	10	1	10	15	9
	11	23	34	30	31
	12	62	62	57	60
	13	98	105	108	104
	14	135	129	119	128
	15	88	87	106	94
	16	27	64	16	36

TABLE III-3

Thermal Resistivity Measurements of Wet Silica Sand

of Moisture Content 9.5%

	MEASUREMENT NUMBER (§3.7)	TRIAL 1 (TRU)	TRIAL 2 (TRU)	TRIAL 3 (TRU)	AVERAGE (TRU)
HEATING PERIOD	1	-67	-108	-80	-85
	2	-46	-76	-66	-63
	3	-14	-40	-54	-36
	4	30	-16	-34	-7
	5	22	-40	-55	-24
	6	-33	-51	-19	-34
	7	-121	-54	31	-48
	8	-139	75	254	63
COOLING PERIOD	10	9	6	5	7
	11	41	30	33	35
	12	57	58	56	57
	13	126	111	94	110
	14	132	93	83	103
	15	147	74	35	85
	16	158	95	-22	77
CORRECTED COOLING PERIOD DATA	10	9	6	5	7
	11	41	30	33	35
	12	58	59	57	58
	13	132	116	98	115
	14	144	101	90	112
	15	173	87	41	100
	16	214	129	-30	104

TABLE III-4

Scatter of Measurements (see text)

MEASUREMENT NUMBER	TRIAL 1	TRIAL 2
1	9	0
2	-6	4
3	-6	-3
4	-12	-6
5	24	4
6	-8	-13
7	-11	-14
8	30	42
9	-25	-34
10	3	5
11	-2	-4
12	2	7
13	-9	-15
14	9	7
15	-6	-25
16	22	-8